Qualitative Reasoning about Economic Dynamics

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July 4, 2006

Abstract. The book presents a theory of boundedly rational reasoning about dynamic economic systems. The theory is inspired by the literature on qualitative physics (Bobrow 1984), but it adds new developments. The aim is a formal reconstruction of verbal qualitative reasoning about economic dynamics. Hume's specie-flow mechanism and Hawtrey's monetary business cycle provide illustrative examples.

A formal definition of a qualitative dynamic system is given. It involves variables with only finitely many values like "high" and "low". The movement of variables in time is described by "tendencies" with only three possible values, $+$ (increasing), 0 (steady) and $-$ (decreasing). Algebraic relationships connect tendencies to variables and other tendencies. Another system part is the "priority assignment" which ranks causal reasons for a transition to a new state.

A qualitative dynamic system permits only finitely many states. A transition from one state to the next involves a chain of causal reasoning formalized as a "readjustment process". The properties of this centrally important algorithm are discussed in detail.

A definition of stability in a qualitative dynamic system is presented. The stationary state is stable in Hume's specie-flow mechanism and instable in Hawtrey's business cycle.

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CHAPTER 1

Introduction

The purpose of this book is a formal reconstruction of the style of verbal reasoning found in the older business cycle literature. This kind of thinking is best exemplified by the famous volume on "prosperity and depression" by von Haberler (1937). Undoubtedly, verbal business cycle theory reached a point of culmination with von Haberler's overview and synthesis.

It is the opinion of the author of this book, that the old style of verbal reasoning about economic dynamics is not just an inferior version of modern quantitative modelling and analysis, but something entirely different, which merits to be studied in detail. It seems to be worthwhile to reconstruct the mental model (in the sense of Johnson Laird or Gentner) and the heuristic principles of analysis underlying pre-mathematical business cycle theory.

This book presents a reconstruction which takes the form of an algorithmic theory. However it is not claimed that this theory adequately reflects reality. This may or may not be the case. The theory is proposed as a formalization of boundedly rational causal reasoning about economic dynamics. It is meant to be a contribution to the emerging field of bounded rationality. However, this does not exclude the possibility, that it may be useful as a tool of analysis.

The reconstruction proposed here is inspired by a book edited by Bobrow (1984) on "qualitative reasoning about physical systems". The author of this book found the articles of de Kleer and Brown and of Kuipers (both in Bobrow (1984)) most helpful for his own thinking. However it turned out to be necessary to add further ideas and to put them into a new framework.

What is qualitative reasoning? Consider a statement of the following kind: "an increase of x causes an increase of y" or "an increase of x causes a decrease of y". Such statements are "qualitative" in the sense that they assert causal connections between directions of change. Nothing is said about the strength of the effect. Qualitative reasoning can be roughly described as reaching qualitative conclusions directly from qualitative assumptions.

The way in which the term is understood in this book emphasizes the word "directly". One can reach some qualitative conclusions from qualitative assumptions by rigorous arguments based on continuity and differentiability requirements imposed on the underlying quantitative system. This is definitely not the aim of this book. Qualitative reasoning as it is understood here avoids the intermediate step of arguing about quantitative models. Instead of this it is guided by heuristic principles which are applied directly without questioning their quantitative validity.

In the literature on qualitative reasoning and related subjects (e.g. Fishwick and Luker (1991), Faltings and Struss (1992), Kuipers (1994)) the term is not always used in the same way as here. There are many different approaches to the subject matter. No attempt will be made to provide an overview.

The theory developed here is based on a precise definition of a qualitative dynamic system and an algorithm for drawing conclusions about how such systems develop over time. Pre-mathematical reasoning about economic dynamics is reconstructed as a mathematical formalism. At some points it will be necessary to supply proofs.

An introduction ordinarily gives a preview of the results. However, an informal description of the content would not be informative for most of the readers who must be expected to be totally unfamiliar with the subject matter. Therefore a preview of the results will not be given here.

The second chapter explains some basic concepts and provides illustrative examples. It begins with the description of a qualitative model of Hume's specie flow mechanism. This model is extremely simple and therefore is well suited for providing a first impression of the approach developed here. However, most of the basic concepts of the theory proposed here cannot be explained with the help of the model for Hume's specie flow mechanism. Therefore a very simple business cycle model will be used as an expositionary device. Later a modification of this model will also be looked upon.

Hawtrey's business cycle theory as described by von Haberler (1937) will not be discussed in the first chapter, but only in chapter 7 of this book, after the instruments for modeling qualitative dynamic systems and the methods for analyzing them will have been fully explained. The simple business cycle models introduced for expositionary purposes are unrelated to Hawtrey's theory.

Chapter 9 will discuss the question how the theory developed here relates to some other approaches to qualitative reasoning about dynamic systems. Chapter 10 will present some concluding remarks.

CHAPTER 2

Basic concepts and illustrative examples

2.1. Hume's specie-flow mechanism

Qualitative reasoning deals with qualitative variables. A **qualitative variable** can only take a finite number of values. These values are ranges like "low" or "high" or border points of such ranges like "capacity limit" in the case of production. The **tendency** of a variable is its direction of change. A tendency can take only three values: $-$ (decreasing), 0 (steady) or $+$ (increasing).

A **confluence** is the qualitative analogue of a differential equation. Each tendency has its confluence. The tendency is on the left hand side and the right hand side connects it to other tendencies and to values of qualitative variables.

In order to illustrate the concept of a confluence we shall explain how Hume's famous specie-flow mechanism (Hume (1752)) can be described by a system of confluences. Hume looks at an open economy in which most commodities are non-traded. Therefore domestic prices can be different from world market prices. The economy is assumed to be small in the sense that it does not have any influence on world market prices. Assume that trade is balanced and that there is a temporary exogenous inflow of gold. Then domestic demand is increased, domestic prices rise, imports go up and exports go down and this results in a trade deficit. As long as the trade deficit persists, gold flows out of the country until trade is balanced again. The case of a temporary outflow of gold is analogous.

It is now necessary to introduce some notations. Variables are represented by strings of capital letters.

TR	trade balance this variable can take the values D deficit b balanced S surplus
GO	gold, the total amount of gold in the country
DE	domestic demand
PR	prices
EX	exports
IM	imports

The trade balance TR is the only variable which can take more than one value. Such variables are called **scaled**, since the values of a scaled variable form a **scale** like D, b, S in the case of TR. The scale lists the values in algebraically increasing order from left to right. Variables with only one possible value are called **unscaled**.

If XY is a variable, then ∂XY denotes the tendency of XY. The use of the symbol "∂" reminds us of the interpretation of a tendency as the sign of a time derivative. The system of confluences for Hume's specie-flow mechanism is as follows:

$$\begin{aligned}
\partial GO &= f(TR) = \begin{cases} - & \text{for} \quad TR = D \\ 0 & \text{for} \quad TR = b \\ + & \text{for} \quad TR = S \end{cases} \\
\partial DE &= \partial GO \\
\partial PR &= \partial DE \\
\partial IM &= \partial PR \\
\partial EX &= -\partial PR \\
\partial TR &= \partial EX - \partial IM
\end{aligned}$$

The interpretation of the first four confluences is straightforward, but the last two require some algebra of directions. A **direction** is one of the possible values $-$, 0 or $+$ of a tendency. If d is a direction then $-d$ is defined as follows.

$$-d = \begin{cases} + & \text{for} \quad d = - \\ 0 & \text{for} \quad d = 0 \\ - & \text{for} \quad d = + \end{cases}$$

The confluence for ∂TR requires that ∂TR is the **sum** of ∂EX and $-\partial IM$. Consider the sum $Z = d_1 + d_2$ of two directions d_1 and d_2. The interpretation of this sum will now be discussed. Consider a functional relationship between three quantitative variables x, y, z:

$$z = f(x, y)$$

Let $\dot{x}, \dot{y}, \dot{z}$ be the time derivatives of x, y, and z. Then we have

$$\dot{z} = \frac{\partial f(x,y)}{\partial x}\dot{x} + \frac{\partial f(x,y)}{\partial y}\dot{y}$$

Suppose that d_1 is the sign of $\dot{x}$ and d_2 is the sign of $\dot{y}$. Moreover, assume that the partial derivatives $\partial f(x,y)/\partial x$ and $\partial f(x,y)/\partial y$ are positive. Then $d_1 + d_2$ is the direction of $\dot{z}$. If, for example $\partial f(x,y)/\partial x$ is positive and $\partial f(x,y)/\partial y$ is negative, then the direction of $\dot{z}$ is $d_1 - d_2$. Algebraic sums of directions like $d_1 - d_2$ are interpreted in this way.

Of course, if one of the directions in the sum $d_1 + d_2$ is positive and the other one negative, nothing can be said about the tendency of the sum. This is expressed by $Z = \{-, 0, +\}$. Table 1 shows the sum Z of two directions d_1 and d_2.

		d_2		
		$-$	0	$+$
d_1	$-$	$-$	$-$	$\{-,0,+\}$
	0	$-$	0	$+$
	$+$	$\{-,0,+\}$	$+$	$+$

TABLE 1. The sum $Z = d_1 + d_2$ of two directions d_1 and d_2

A graphical representation of the model is shown by Figure 1.

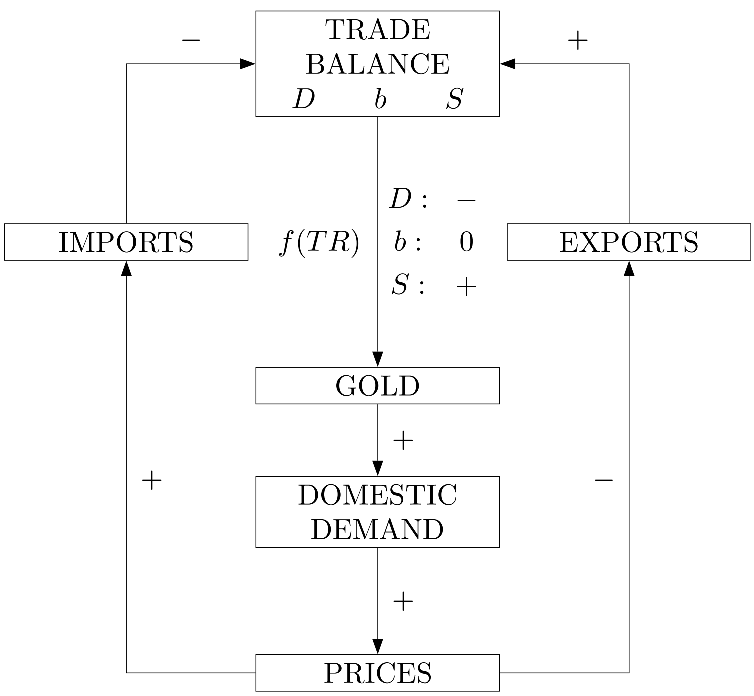


FIGURE 1. Hume's specie-flow mechanism

The variables appear in rectangles and the causal influences expressed by the confluences are shown by connecting lines with arrows indicating the direction of causation. A "+" or a "−" at such a line shows whether the influence is positive or negative. It may happen, however, that the sign of an influence depends on values of variables as in the case of the confluence for ∂GO.

In the model of Hume's specie-flow mechanism $-\partial EX$ and ∂IM are both equal to ∂PR. Therefore we have

$$\begin{aligned} \partial TR &= \partial EX - \partial IM \\ &= -\partial PR - \partial PR \\ &= -\partial PR \\ &= -\partial DE \\ &= -\partial GO \\ &= -f(TR). \end{aligned}$$

The value of TR unambiguously determines ∂TR.

2.2. States and transition diagram

For a given specification of all values of variables the right hand side of a confluence will in general be a set of directions. A confluence is **satisfied** if the tendency on the left hand side is an element of this set. Of course, in special cases the right hand side is a single direction and then a confluence is satisfied if both sides are equal.

For the sake of simplicity no distinction is made between a set with only one element and this element. Admittedly in confluences the equality sign does not really have the meaning of asserting equality, but rather that of the set theoretic sign $\in$. Therefore left and right hand sides cannot be interchanged. Once this is understood, no misunderstandings can arise from the usual notational conventions concerning confluences.

A **state** of a system of confluences is a specification of the values of all scaled variables and of the tendencies of all variables, such that all confluences are satisfied. This definition of a state is preliminary. Later it will have to be a adjusted to more complex systems. The final definition of a state will be given at the end of Section 2.7. The model for Hume's specie-flow mechanism has exactly three states since all tendencies are uniquely determined by the value of TR.

state	TR	∂TR
1	D	$+$
2	b	0
3	S	$-$

If the system is in state 1 then the trade balance will improve in view of $\partial TR = +$ until trade becomes balanced. Therefore state 2 will be reached from state 1. Analogously state 3 leads to the state 2, too. Figure 2 describes these conclusions in the form of a **transition diagram.**

FIGURE 2. The transition diagram for Hume's specie-flow mechanism

Of course, the argument about the transitions from states 1 and 3 to state 2 is heuristic rather than exact. A decreasing positive variable may never reach zero, since it may converge to a positive asymptote. Such possibilities are ignored by qualitative reasoning in the sense in which the term is used in this book. However this should not be considered to be a mistake, but rather an implicit assumption about the underlying quantitative system.

It can be seen that state 2 is stable by any reasonable definition of the term. A small disturbance may move the system temporarily to state 1 or state 3, but from there it must return to state 2.

The model of Hume's specie-flow mechanism is exceptionally simple. It is easy to construct a transition diagram and to investigate the stability of the stationary state. It is much more difficult to develop a general method for solving the same problems for a broad class of qualitative dynamic systems.

2.3. Boundary restrictions

The notion of a confluence is common to most of the literature on qualitative reasoning. However, the adequate representation of verbal business cycle theories seems to require additional conceptual instruments. Therefore the theory proposed here makes use of a restriction concept.

Consider a qualitative variable "production" denoted by PD with the scale b, L, n, H, c. We think of production as the total output of an economy. The symbol b denotes a lower limit, below which production cannot fall for technological or social reasons. L and H stand for ranges of low and high production, n (normal) is the border point between L and H and c is the capacity limit.

On a scale **points** like b, n, c alternate with **ranges** like L and H. Ranges are interpreted as open intervals of an underlying quantity and points as border points of such intervals. We use capital letters for ranges and lower case letters for points. A scale has a **bottom value** at its lower end and a **top value** at its upper end. If the bottom value is a point it is called a **bottom point**. Similarly a **top point** is a top value which is a point. However, top or bottom values may also be ranges. In this case we speak of **top ranges** and **bottom ranges**. A scaled variable with a top point or a bottom point is called **bounded**. This is suggested by the idea that an underlying quantitative variable would have to be bounded from below or above.

At the capacity limit c production cannot be further increased. Therefore at c the tendency ∂PD must be in $\{-,0\}$. This set is the boundary restriction of ∂PD at c. Similarly $\{0,+\}$ is the boundary restriction of ∂PD at b. At L, n, and H the tendency ∂PD is not constrained by a boundary restriction. This is expressed by saying that there the boundary restriction is the set $\{-,0,+\}$ of all possible directions.

The symbol $\rhd$ followed by the name of the variable denotes the boundary restriction of this variable. Let XY be a scaled variable: Then we have

$$\rhd XY = \begin{cases} \{-,0\} & \text{for the top point of } XY\text{, if } XY \text{ has one} \\ \{0,+\} & \text{for the bottom point of } XY\text{, if } XY \text{ has one} \\ \{-,0,+\} & \text{else.} \end{cases}$$

The boundary restriction of a tendency may appear in its confluence. In order to explain this in detail we need some explanations about the algebra of convex direction sets which will follow in the next section.

2.4. The algebra of convex direction sets

A **direction set** is a non-empty subset of $\{-,0,+\}$. We call $\{-,0,+\}$ the **full** direction set. A **convex** direction set is characterized by the condition that zero must be in it if $+$ and $-$ belong to it. The direction set $\{-,+\}$ is the only one excluded by this definition. For the sake of simplicity we make no distinction between a direction and the convex direction set containing this direction as its only element. There are altogether 6 convex direction sets: $-, 0, +, \{-,0\}, \{0,+\}$, and $\{-,0,+\}$.

We now extend the definition of a sum to convex direction sets. Let $S_1, ..., S_n$ be convex direction sets. Then the **sum**

$$S = S_1 + ... + S_n$$

of these sets is defined as follows:

S contains $-$	if and only if $-$ is in one of the sets $S_1, ..., S_n$
S contains $+$	if and only if $+$ is in one of the sets $S_1, ..., S_n$
S contains 0	if and only if $-$ and $+$ belong to S or 0 belongs to each of the sets $S_1, ..., S_n$

In an algebraic sum some components may appear with a negative sign. The subtraction of S_i is defined as the addition of $-S_i$. A direction d belongs to $-S_i$ if and only if $-d$ belongs to S_i.

The sum of two convex direction sets is shown by Table 2. The interpretation of a sum of convex direction sets is similar to that of a sum of directions. One thinks of each of the S_k with $k = 1, ..., n$ as connected to an underlying quantity whose tendency is in S_k. The sum S is the set of all possible tendencies of the sum of the time derivatives of these quantities multiplied with the relevant partial derivatives (see 2.1).

		S_2					
		$-$	0	$+$	$\{-,0\}$	$\{0,+\}$	$\{-,0,+\}$
S_1	$-$	$-$	$-$	$\{-,0,+\}$	$-$	$\{-,0,+\}$	$\{-,0,+\}$
	0	$-$	0	$+$	$\{-,0\}$	$\{0,+\}$	$\{-,0,+\}$
	$+$	$\{-,0,+\}$	$+$	$+$	$\{-,0,+\}$	$+$	$\{-,0,+\}$
	$\{-,0\}$	$-$	$\{-,0\}$	$\{-,0,+\}$	$\{-,0\}$	$\{-,0,+\}$	$\{-,0,+\}$
	$\{0,+\}$	$\{-,0,+\}$	$\{0,+\}$	$+$	$\{-,0,+\}$	$\{0,+\}$	$\{-,0,+\}$
	$\{-,0,+\}$	$\{-,0,+\}$	$\{-,0,+\}$	$\{-,0,+\}$	$\{-,0,+\}$	$\{-,0,+\}$	$\{-,0,+\}$

TABLE 2. The sum S of two convex direction sets S_1 and S_2

It can be seen without difficulty that the addition of convex direction sets is commutative and associative. Moreover, it is clear that a sum of convex direction sets is a convex direction set. Zero belongs to it if + and − are in it.

A **direction sum** is a direction set which can be obtained as the sum of directions. Let $d_1, ..., d_n$ be directions. Consider the sum

$$D = d_1 + ... + d_n$$

Obviously the directions $-, 0, +$ are direction sums, $\{-, 0, +\}$ is the sum of $-$ and $+$ and therefore is a direction sum, too. However the remaining two convex direction sets $\{-, 0\}$ and $\{0, +\}$ fail to be direction sums. Zero can be in D if a $+$ and a $-$ is among the d_k but in this case D equals $\{-, 0, +\}$. Otherwise zero can be in D only if all d_k are equal to zero, but then D is zero. Therefore $-, 0, +$ and $\{-, 0, +\}$ are the only direction sums.

The notion of a direction sum is important for the structure of confluences. A confluence for a tendency represents the combined influences of other variables by an algebraic sum of other tendencies, maybe augmented by a constant direction. This expression on the right hand side is the **main term** of the confluence. The main term may depend on values of scaled variables but for given values of all scaled variables it has this structure. Therefore at a state the value of a main term must be a direction sum.

It may happen, however, that a tendency is subject to a boundary restriction. In this case the main term must be modified in order to take account of the restriction. If the intersection of the main term with the restriction is non-empty then this intersection is the value of the right hand side. However this is different, if the intersection is empty.

It is useful to look at an example. Suppose that production is represented by the variable PD with the scale b, L, n, H, c. Assume that usually production quickly adjusts to real effective demand, an unscaled variable DE. This means that $\partial PD = \partial DE$ holds unless the boundary condition $\triangleright PD$ is binding.

Consider the case $\partial DE = +$ and $PD = c$. In this case we have $\triangleright PD = \{-, 0\}$. Rising demand pushes production up, but the capacity limit stops its upward movement, like the rise of a gas filled toy balloon is stopped by the ceiling. Therefore we have $\partial PD = 0$ for $\partial DE = +$ and $PD = c$. This is formally described by the operation of **accomodation** expressed by the symbol @. Let d be a direction and let R be a convex direction set. d @ R, read as "d accommodated to R", is the element of R **nearest** to d in the following sense: $+$ and $-$ are **nearer** to 0 than to each other. Of course, d is **nearer** to itself than to any other direction.

In the same way as a direction d, a convex direction set S can be accommodated to a convex direction set R. The expression S @ R is defined as follows

$$S \mathbin{@} R = \begin{cases} S \cap R & \text{if} \quad S \cap R \neq \emptyset \\ R & \text{if } R \text{ contains only one element} \\ 0 & \text{if } S = - \text{ and } R = \{0, +\} \\ 0 & \text{if } S = + \text{ and } R = \{-, 0\} \end{cases}$$

Table 3 shows S @ R for any two convex direction sets R and S.

		R					
		$-$	0	$+$	$\{-, 0\}$	$\{0, +\}$	$\{-, 0, +\}$
S	$-$	$-$	0	$+$	$-$	0	$-$
	0	$-$	0	$+$	0	0	0
	$+$	$-$	0	$+$	0	$+$	$+$
	$\{-, 0\}$	$-$	0	$+$	$\{-, 0\}$	0	$\{-, 0\}$
	$\{0, +\}$	$-$	0	$+$	0	$\{0, +\}$	$\{0, +\}$
	$\{-, 0, +\}$	$-$	0	$+$	$\{-, 0\}$	$\{0, +\}$	$\{-, 0, +\}$

TABLE 3. The value of S @ R for two convex direction sets S and R

With the help of the accommodation operation the relationship between ∂PD and ∂DE assumed above can now be expressed as follows

$$\partial PD = \partial DE \;@\; \rhd PD.$$

In this way the main term of a confluence for the tendency of a scaled variable with a top point or a bottom point is accommodated to the boundary restriction of the tendency.

2.5. System specific restrictions and restriction equations

Boundary restrictions are a simple consequence of the scale of the concerning variable. They are independent of other aspects of the specific system. However, a tendency may be restricted in another way which needs to be modeled explicitly. This will lead us to system specific restrictions and restriction equations. These concepts will be explained in the context of a very simple business cycle model. In fact this model is too simple to be taken seriously, but it is useful as an expositional device.

In addition to the variable PD, production, with the scale $b, L, n,\ H, c$ the model contains the unscaled variables DE, real effective demand and IN, the rate of inflation. It is assumed that above the "normal" level n of PD an overuse of productive resources results in increasing inflation. Similarly the inflation rate decreases below n. Only at n it is steady. This leads to the following confluence for ∂IN:

$$\partial IN = \begin{cases} - & \text{for} \quad PD = b, L \\ 0 & \text{for} \quad PD = n \\ + & \text{for} \quad PD = H, c \end{cases}$$

The confluence for ∂DE is based on the idea that real income and therefore real effective demand are positively influenced by production. Ceteris paribus a rising rate of inflation decreases real income and thereby real effective demand. This leads to the main term $\partial PD - \partial IN$ in the confluence for ∂DE. This main term is accommodated to a system specific restriction $\Box DE$

$$\partial DE = (\partial PD - \partial IN) \;@\; \Box DE$$

The symbol $\Box$ is used analogously to $\rhd$. A system specific restriction is denoted by $\Box$ followed by the name of the variable whose tendency is restricted. System specific restrictions need to be modeled explicitly by restriction equations. The restriction equation for $\Box DE$ is very simple

$$\Box DE = \rhd PD$$

It is assumed that inventories serve transactional purposes only and therefore are kept constant. Accordingly effective demand cannot grow any more once the capacity limit has been reached. At the lower bound b of production DE cannot fall since always the whole production will be sold, if necessary at sufficiently low prices. Usually production follows effective demand, but the boundary restriction $\triangleright PD$ may constrain it. As in 2.3 we have:

$$\partial PD = \partial DE \ @ \ \triangleright PD$$

Table 4 summarizes the simple business cycle model.

Variables

PD	production, scale b, L, n, H, c
IN	rate of inflation, unscaled
DE	real effective demand, unscaled

Confluences

$$\partial PD = \partial DE \ @ \ \triangleright PD$$

$$\partial IN = \begin{cases} - & \text{for} \quad PD = b, L \\ 0 & \text{for} \quad PD = n \\ + & \text{for} \quad PD = H, c \end{cases}$$

$$\partial DE = (\partial PD - \partial IN) \ @ \ \Box DE$$

Restriction equation

$$\Box DE = \triangleright PD$$

TABLE 4. A simple business cycle model

As has been explained before, a confluence is satisfied, if the left hand side is an element of the right hand side. However, a restriction equation is **satisfied**, if the sets on the left hand side and the right hand side are equal.

A preliminary definition of a state has been presented in Section 2.2. This definition must be adjusted to the presence of system specific restrictions. A **state** is a specification of values for all scaled variables, for the tendencies of all variables and for all system specific restrictions, such that all confluences and restriction equations are satisfied. The value of a scaled variable is on its scale, the values of tendencies are directions and the values of system specific restrictions are convex direction sets. This definition of a state is still preliminary. The final definition will be given at the end of Section 2.7.

The structure of the right hand side of a restriction equation may be more complex than in our example. There is always a **main term** S which is a sum of convex direction sets which may involve tendencies or restrictions of other variables or constant direction sets. In the case of a bounded variable (a scaled variable with a bottom point or a top point, see 2.3) the main term S must be accommodated to the boundary restriction:

$$\Box XY = S \text{ @ } \rhd XY$$

The main term may depend on values of variables, but if they are given, it is a sum of convex direction sets as described above.

2.6. States and cycle of the simple model of Table 4

We continue to look at the simple business cycle model of Table 4. This model has only 9 states. They are listed in Table 5. State 9 is the only one with $\partial PD = 0$. This can be seen as follows. If ∂PD is zero then it follows by the confluences for ∂DE and ∂IN that we have

$$\partial PD = \begin{cases} + & \text{for} \quad PD = b, L \\ 0 & \text{for} \quad PD = n \\ - & \text{for} \quad PD = H, c \end{cases}$$

This shows that $\partial PD = 0$ can hold only at $PD = n$. All confluences are satisfied for $PD = n$ and

$$\partial PD = \partial IN = \partial DE = 0$$

Everywhere else we must have $\partial PD = -$ or $\partial PD = +$. Therefore at $PD = b$ we must have $\partial PD = +$ in view of $\rhd PD = \{0, +\}$ and at $PD = c$ we must have $\partial PD = -$ in view of $\rhd PD = \{-, 0\}$. This means that state 1 is the only one with $PD = b$ and state 5 is the only one with $PD = c$. It follows by the confluence for ∂PD that we always have

$$\partial PD = \partial DE$$

For $PD = L, n, H$ the value of ∂PD can be $+$ or $-$. This leads to states $2, 3, 4$ and $6, 7, 8$ respectively. This shows that there cannot be any other states than those listed in Table 5 and at each of these states all confluences are satisfied.

The question arises how the system moves from one state to the other. At the moment we can only give a preliminary answer. We proceed from the heuristic principle that no more is changed in the transition than is necessary. In the case of the confluences for the model of Hume's specie-flow mechanism the state of the system was completely determined by the value of TR. Therefore the movements from state to state depended only on ∂TR. The situation is a little more complex in the simple business cycle model.

state	PD	$\triangleright PD = \Box DE$	∂PD	∂DE	∂IN
1	b	$\{0, +\}$	$+$	$+$	$-$
2	L	$\{-, 0, +\}$	$+$	$+$	$-$
3	n	$\{-, 0, +\}$	$+$	$+$	0
4	H	$\{-, 0, +\}$	$+$	$+$	$+$
5	c	$\{-, 0\}$	$-$	$-$	$+$
6	H	$\{-, 0, +\}$	$-$	$-$	$+$
7	n	$\{-, 0, +\}$	$-$	$-$	0
8	L	$\{-, 0, +\}$	$-$	$-$	$-$
9	n	$\{-, 0, +\}$	0	0	0

TABLE 5. The 9 states of the simple business cycle model of Table 4

Consider state 1. There we have $\partial PD = +$. This means that PD moves towards L. However, there are two states with $PD = L$, namely state 2 and state 8. The movement of PD from b to L alone does not determine the next state. It is important that nothing else needs to be changed. At state 2 all tendencies have the same value as at state 1. This is not true for state 8. Therefore the next state after state 1 is state 2.

As we shall see later, a transition is initiated by a **transition cause**, in our case by a change of the value of PD from b to L. A change of the value of a scaled variable to the next higher or the next lower value is called a **shift**. No other transition causes are considered in this section.

A shift from a point to a range is **immediate** in the sense that it must happen without delay. Since ∂PD is positive at state 1, the system cannot stay there for more than a moment. PD must move from b to L without any delay. A shift from a range to a point is called **tardy**. A scaled variable may stay in a range for a long time, even if eventually it must move to a point.

The distinction between immediate and tardy transition causes is important for the theory proposed here. Consider a system with two scaled variables. Suppose that an immediate shift of one of them and a tardy shift of the other one are possible at the same state. Then the immediate shift has absolute priority. It must happen before the tardy shift has any chance to become effective. Of course, this situation cannot arise in the simple business cycle model. The only scaled variable in this model is PD.

At state 2 we have $PD = L$ and $\partial PD = +$. Therefore a tardy shift of PD from L to n is a transition cause at state 2. For $PD = n$ the value of ∂IN must change from $-$ to 0 in order to satisfy the confluence for ∂IN, but after this change all confluences are satisfied. The tardy shift from L to n leads to state 3. In the same way an immediate shift from n to H leads from state 3 to state 4.

$\partial PD = +$ still holds at state 4. Therefore a tardy shift of PD from H to c must take place at state 4. This transition cause must lead to state 5, since $PD = c$ holds at no other state.

States 1 to 8 form the cycle of the simple business cycle model. This cycle is graphically described by Figure 3. The transitions from state 1 to state 5 form the upswing of this cycle. The downswing from state 5 to state 8 and from there back to state 1 is analogous to the upswing. It is not necessary to discuss this in detail.

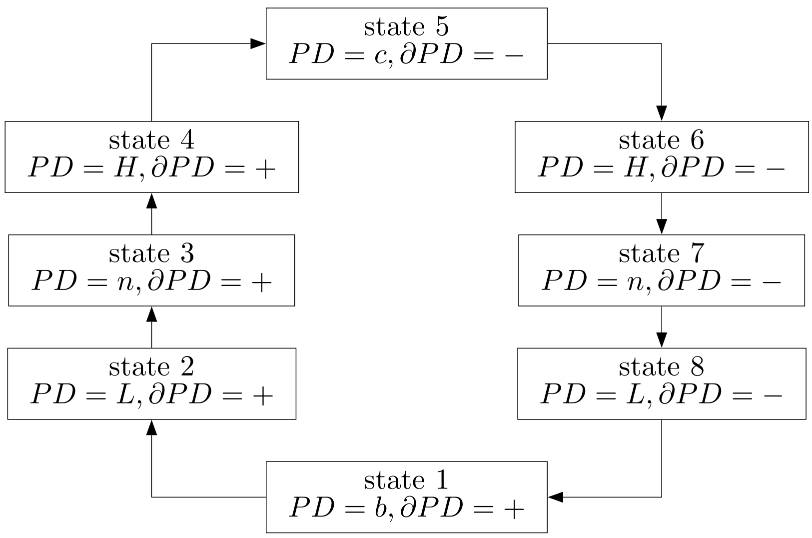


FIGURE 3. The cycle of the simple business cycle model

Transitions are initiated by transition causes, but where a transition cause leads to is determined by a readjustment process to be explained in chapter 4. This process is an algorithm which adjusts unsatisfied confluences and restriction equations until a new state is reached. In this section it is not yet possible to discuss the readjustment process. What has been said about the cycle of the simple business cycle model was based on heuristic arguments. The general idea was that in a transition only necessary changes should be made. However, this somewhat imprecise "principle of minimal change" does not fully reflect the properties of the readjustment process.

2.7. Lagged tendencies, final state definition and system base

A confluence or a restriction equation may express a dependence on the value of a tendency in the recent past. This gives rise to the notion of a **lagged tendency**.

The lag is indicated by the superscript "$-$". Thus ∂XY^- is the lagged tendency of the variable XY. In order to avoid confusion, we shall speak of the tendency ∂XY as the **current tendency** where this is necessary. The word "tendency" without the qualification "lagged" will refer to a current tendency. However, we also speak of "current and lagged tendencies" instead of "current tendencies and lagged tendencies".

A modified version of the simple business cycle model of Table 4 shown by Table 6 provides an example for a confluence with a lagged tendency on the right hand side. Apart from the confluence for ∂DE the modified model agrees with the original one. In this confluence ∂PD is replaced by ∂PD^-.

Variables

PD	production, scale b, L, n, H, c
IN	rate of inflation, unscaled
DE	real effective demand, unscaled

Lagged tendency

∂PD^-

Confluences

$$\partial PD = \partial DE \ @ \ \triangleright PD$$

$$\partial IN = \begin{cases} - & \text{for} \quad PD = b, L \\ 0 & \text{for} \quad PD = n \\ + & \text{for} \quad PD = H, c \end{cases}$$

$$\partial DE = (\partial PD^- - \partial IN) \ @ \ \Box DE$$

Restriction equation

$$\Box DE = \triangleright PD$$

Table 6. The modified simple business cycle model

Suppose that a lagged tendency and a current tendency have different values. If ∂XY does not change, then after a while the time when ∂XY had the lagged value will not any more be in the recent past. This means that then ∂XY^- will have to change its value to that of the current tendency ∂XY. Such a **lag extinction** is bound to happen sooner or later if the values of ∂XY^- and ∂XY are different and ∂XY does not change.

Having introduced the notion of a lagged tendency we are now ready to give the final definition of a state. A specification of values for all scaled variables, for all current and lagged tendencies and for all system specific restrictions is **admissible**, if the following three conditions are satisfied:

(a1) Each scaled variable has a value on its scale.
(a2) The values of current and lagged tendencies are directions.
(a3) The values of system specific restrictions are convex direction sets.

A **state**, is an admissible specification of values for all scaled variables, all current and lagged tendencies and all system specific restrictions with the following additional property:

(a4) All confluences and restriction equations are satisfied for the specified values.

The models considered up to now have a common structure. This structure has two parts. The first part is a **list of variables** Λ involving scaled variables with their scales and unscaled variables. The number of variables in the list is finite and not zero. The second part is a **list** Γ **of confluences and restriction equations**. This list Γ must **fit** the list Γ of variables in the sense of the following three conditions:

(b1) The list Γ contains one and only one confluence for each current tendency for a variable in Λ.
(b2) The list Γ contains one and only one restriction equation for each system specific restriction appearing on the right hand side of a confluence and no other restriction equations.
(b3) All current and lagged tendencies and all boundary or system specific restrictions appearing on the right hand side of confluences and restriction equations belong to variables in Λ. Moreover only system specific restrictions with restriction equations in Γ appear on the right hand side of other restriction equations.

We call a pair $B = (\Lambda, \Gamma)$ of this kind a **system base** or shortly a **base**. However, (b1), (b2), and (b3) do not yet exhaust the description of Γ. Additional conditions will be imposed on confluences and restriction equations and on the list Γ as a whole. Only after this will have been done the final definition of a system base can be given in 2.12.

A system base is not yet a full-fledged qualitative dynamic system. The definition of a qualitative dynamic system will be given at the beginning of chapter 4. This definition involves two further parts in addition to Λ and Γ. It will become clear in chapter 3 why the base is not yet a sufficient description of a qualitative dynamic system.

In Table 6 the lagged tendency ∂PD^- is explicitly listed. This is not really necessary, since the confluences and restriction equations show which lagged tendencies are present. Therefore a base does not contain a separate list for lagged tendencies. The model of Table 6 will be further examined in the next section.

2.8. States and cycle of the model of Table 6

In order to determine the states of the model of Table 6 we look at the possibilities for ∂DE. The values of the right hand side of the confluence for ∂DE as a function of ∂PD^- and PD are shown by Table 7.

		PD				
		b	L	n	H	c
∂PD^-	$-$	$\{0,+\}$	$\{-,0,+\}$	$-$	$-$	$-$
	0	$+$	$+$	0	$-$	$-$
	$+$	$+$	$+$	$+$	$\{-,0,+\}$	$\{-,0\}$

TABLE 7. Values of the right hand side of the confluence for ∂DE in the modified simple business cycle of Table 6

There are exactly 21 possibilities for triples of $\partial PD^-, PD$ and ∂DE. In view of $\Box DE = \triangleright PD$ it follows by the confluence for ∂PD that we have $\partial PD = \partial DE$ at every state. Moreover ∂IN is determined by the value of PD. Therefore each of the 21 triples determines exactly one state. The list of all 21 states is shown by Table 8.

In the analysis of the model one has to deal with two kinds of transition causes: shifts and lag extinctions. Sometimes a shift as well as a lag extinction is possible at the same state. This happens at state 3. Here it is reasonable to give priority to the immediate shift of PD from b to L.

A cycle of the model of Table 6 can be constructed on the basis of shifts and lag extinctions. A shift of PD is possible for $\partial PD \neq 0$ and a lag extinction is possible for $\partial PD^- \neq \partial PD$. If the two kinds of transition causes are possible at a state, then priority is given to immediate shifts at point values of PD and to lag extinctions over tardy shifts at range values of PD.

It is clear that priority must be given to immediate shifts at point values, but at range values one could consider some other priority rule. The rule chosen here reflects the idea that the duration of a lag is short in comparison to the time for which PD stays at a range value.

At least implicitly some assumptions on the priorities among different transition causes are made in qualitative reasoning. As we shall see, the notion of a

state	PD	∂PD^-	$\Box DE$	∂IN	∂DE	∂PD
1	b	$-$	$\{0,+\}$	$-$	0	0
2	b	$-$	$\{0,+\}$	$-$	$+$	$+$
3	b	0	$\{0,+\}$	$-$	$+$	$+$
4	b	$+$	$\{0,+\}$	$-$	$+$	$+$
5	L	$-$	$\{-,0,+\}$	$-$	$-$	$-$
6	L	$-$	$\{-,0,+\}$	$-$	0	0
7	L	$-$	$\{-,0,+\}$	$-$	$+$	$+$
8	L	0	$\{-,0,+\}$	$-$	$+$	$+$
9	L	$+$	$\{-,0,+\}$	$-$	$+$	$+$
10	n	$-$	$\{-,0,+\}$	0	$-$	$-$
11	n	0	$\{-,0,+\}$	0	0	0
12	n	$+$	$\{-,0,+\}$	0	$+$	$+$
13	H	$-$	$\{-,0,+\}$	$+$	$-$	$-$
14	H	0	$\{-,0,+\}$	$+$	$-$	$-$
15	H	$+$	$\{-,0,+\}$	$+$	$-$	$-$
16	H	$+$	$\{-,0,+\}$	$+$	0	0
17	H	$+$	$\{-,0,+\}$	$+$	$+$	$+$
18	c	$-$	$\{-,0\}$	$+$	$-$	$-$
19	c	0	$\{-,0\}$	$+$	$-$	$-$
20	c	$+$	$\{-,0\}$	$+$	$-$	$-$
21	c	$+$	$\{-,0\}$	$+$	0	0

TABLE 8. The states of the modified simple business cycle model of Table 6

qualitative dynamic system makes such assumptions explicit as a formal part of the definition.

The cycle produced by shifts and lag extinctions with the priority rule described above is shown by Figure 4. As in the case of the cycle of Figure 3 current tendencies are not changed in the transition from one state to the next unless this is necessary. In the transition from state 1 to state 3 the right hand side of the confluence for ∂DE becomes positive (see Table 7). Therefore ∂DE and ∂PD have to change from 0 to $+$. In the upswing on the left hand side of Figure 4 the tendency ∂PD does not change its value $+$ up to state 17. At state 21 it has to change to 0 in view of the boundary restriction of ∂PD. The transition from state 21 to state 19 is analogous to that from state 1 to state 3.

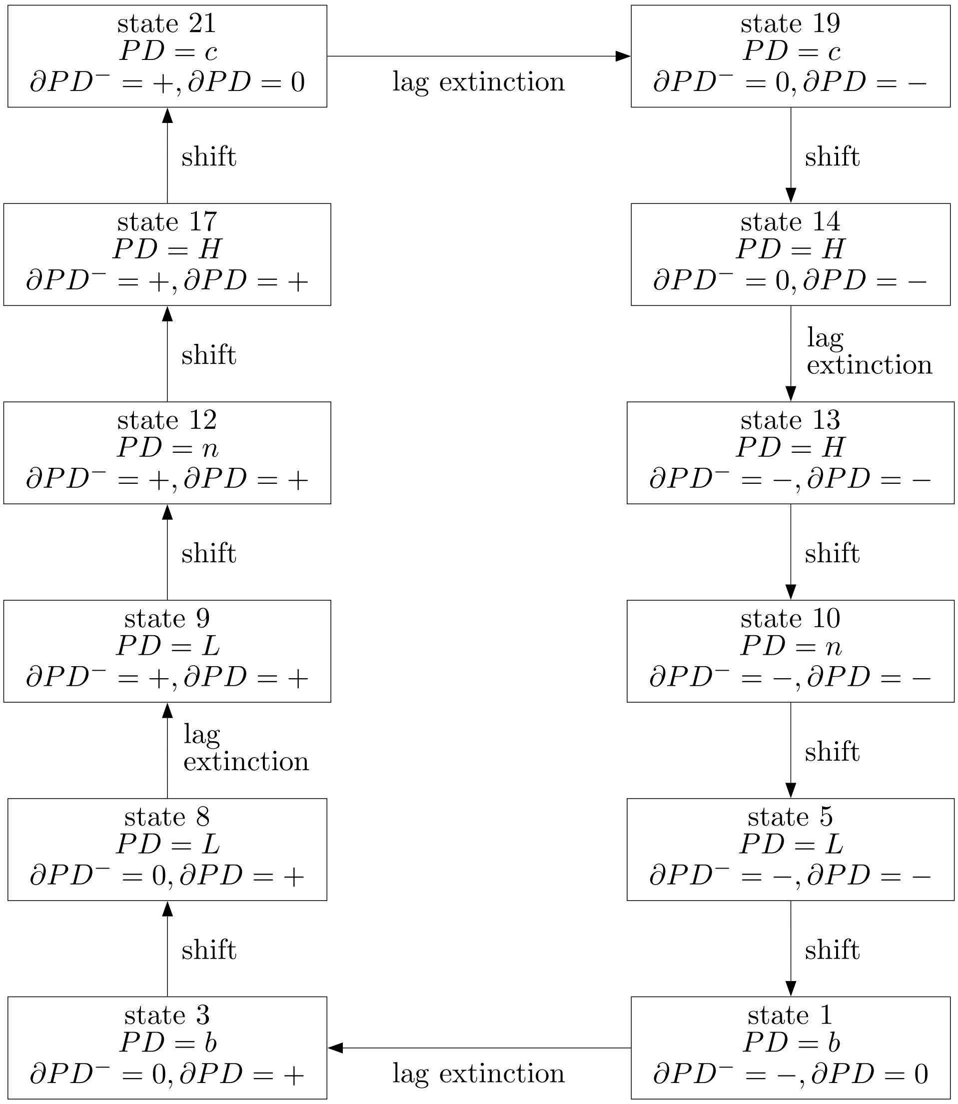


FIGURE 4. The cycle for the model of Table 6

In general the principle of not changing more than necessary is not sufficient for determining a unique result of a transition. At the moment we cannot give more than a heuristic discussion. More precise definitions will be given in the description of the readjustment process in chapter 4.

2.9. Tendency switches

Up to now two kinds of transition causes have been considered, shifts and lag extinctions. In this section a third category of transition causes called "tendency switches" will be introduced. We shall first look at an example. At state 4 of the simple business cycle of Table 4 the tendency ∂DE has the value + and the value of the right hand side of the confluence for ∂DE is $\{-,0,+\}$. This means that at state 4 the positive influence of ∂PD on ∂DE is stronger than the negative influence of ∂IN. However, as PD approaches c this balance of forces may change. The positive influence of ∂PD may become weaker than the negative influence of ∂IN. If this happens ∂DE changes its value from $+$ to $-$. Such a movement of a tendency from a value d_1 to another value d_2 in the right hand side of its confluence is called a **tendency switch** or shortly a **switch**.

The tendency switch of ∂DE from $+$ to $-$ considered above leads from state 4 to state 6. There are other states with $\partial DE = -$, but a transition to state 6 involves a minimum of change. Of course, here too, the principle of minimal change is not more than a heuristic argument. In the theory proposed here, a readjustment process described in chapter 4 determines what happens, if a transition cause becomes effective.

A tendency switch from state 4 to state 6 means that the upswing ends and is followed by a downswing before the capacity limit is reached. In the construction of the cycle of Figure 3 we have ignored this possibility. Priority was given to shifts. Whether this is judged to be plausible or not is a modelling decision which will be formally expressed by a priority ranking in chapter 3.

At state 8 a similar tendency switch from $-$ to $+$ leads to state 2. In this way the downswing may end before the bottom point b is reached.

Of course a movement of a tendency from $-$ to $+$ must go through zero, but it does not have to stop there for more than a moment. Therefore we think of a tendency switch from $-$ to $+$ or from $+$ to $-$ as a single transition rather than a sequence of two transitions.

In section 2.6 a distinction between immediate and tardy shifts has been introduced. A similar distinction has to be made for tendency switches. Consider a state s of a system base $B = (\Lambda, \Gamma)$ and let ∂XY be a tendency of B whose value at s is zero. Moreover assume that the right hand side of the confluence for ∂XY has the value $\{-,0,+\}$. In this situation the positive and negative influences on ∂XY must be exactly balanced. We think of the quantitative time derivatives underlying the tendencies as constantly moving. Therefore, the exact balance cannot be expected to last for more than a moment. ∂XY must change immediately from zero to $-$ or $+$. We refer to such changes as **immediate** tendency switches. Tendency switches which are not immediate are called **tardy**. Tendency switches from $-$ to $+$ or $+$ to $-$ are always tardy. A negative or positive balance of influences on a tendency may persist for a long time.

Switches of a tendency ∂XY are also possible at a state s at which the right hand side of the confluence for ∂XY has the value $\{-,0\}$ or $\{0,+\}$. In these cases a switch of ∂XY may go from $-$ or $+$ to zero or from zero to $-$ or $+$. We refer to such switches as **restricted switches**. As has been explained before (see 2.4) the value of the main term of a confluence is a direction sum, that is, a single direction or the full direction set. In the case of a single direction a tendency switch is impossible. If a tendency switch of ∂XY is possible at a state s, then the main term of the confluence for ∂XY must have the value $\{-, 0, +\}$. Moreover, if the value of the right hand side of the confluence for ∂XY has exactly two elements, then this value must be the value of the restriction $\triangleright XY$ or $\square XY$ of ∂XY at s. Therefore a restricted tendency switch is a switch of ∂XY within a restriction $\triangleright XY$ or $\square XY$.

In the theory proposed here, restricted tendency switches are always considered to be tardy. We refer to this as the **tardiness assumption about restricted switches**. As far as switches from zero to $-$ or $+$ are concerned, this assumption is justified by the idea that a negative or positive balance of the influences on ∂XY within its restriction may last a long time. The situation is less simple for restricted switches from zero to $-$ or $+$. After all such switches are considered to be immediate if they are not restricted. However, in the case of a restricted switch the balance of the influences on the main term may be outside of the restriction, that is, the balance may be positive if this value is $\{-, 0\}$ or negative if it is $\{0, +\}$. Obviously in these cases restricted switches must be regarded as tardy. The tardiness assumption about restricted tendency switches amounts to the idea that in the case of a restriction with the value $\{-, 0\}$ or $\{0, +\}$ the value zero of the restricted tendency always indicates a balance of the influences on the main term outside the value of the restriction. This is plausible, since a negative or positive balance is much more likely than a balance at exactly zero.

Nevertheless in some contexts there may be reasons not to rely on the tardiness assumption. It is always possible to do this by modelling the balance of the influences on the main term as a separate variable. It will be explained in Section 2.11 how this is done.

Table 9 summarizes what has been said about tendency switches. It shows which switches are possible on the basis of the confluence for a single tendency and maybe the restriction equation for its system specific restriction. Actually the system as **?????**. switches. If the bindingness requirement is satisfied the only immediate switches of a tendency ∂XY are switches from zero to $-$ or $+$ at states at which the right hand side of the confluence for ∂XY has the value $\{-, 0, +\}$.

There is an important difference between tendency switches on the one hand and shifts and lag extinctions on the other hand. Values of scaled variables and lagged tendencies remain constant during the readjustment process described in chapter 4 whereas current tendencies can be changed by the readjustment process. Shifts and lag extinctions are transition causes which always lead to a transition to

right hand side of the confluence	Possible switches		immediate or tardy
	From	to	
$\{-, 0, +\}$	$-$	$+$	tardy
	$+$	$-$	tardy
	0	$+$	immediate
	0	$-$	immediate
$\{-, 0\}$	$-$	0	tardy
	0	$-$	tardy
$\{0, +\}$	$+$	0	tardy
	0	$+$	tardy

TABLE 9. Possible tendency switches

a new state, once they become effective. In this sense shifts and lag extinctions are always **feasible**. Tendency switches are transition causes which are not necessarily feasible in the same sense. In Section 3.2.3 the example of a system A will be presented. This system has only one state. However, at this state the value of the right hand side of the confluence for a tendency ∂AA is $\{-, 0, +\}$ and the value of ∂AA is $-$. A tendency switch of ∂AA from $-$ to $+$ is present as a transition cause at the only state of system A, but a transition to a state with $\partial AA = +$ is not feasible, since there is no state with $\partial AA = +$.

Tendency switches can be described as hypothetical transition causes. Feasibility is not guaranteed but must be explored with the help of the readjustment process. More about this will be said in 3.2.

2.10. The structure of confluences and restriction equations

In this section we turn our attention to structural properties required for single confluences and restriction equations of a base $B = (\Lambda, \Gamma)$. These requirements will be expressed by conditions (c1) to (c10).

Confluences and restriction equations may depend on values of scaled variables. The confluence for ∂GO in the model of Hume's specie-flow mechanism and the confluence for ∂IN in the simple business cycle model of Table 4 are examples. Within the limits of the ten conditions the dependence of right hand sides on scale values can be freely specified.

The ten conditions concern confluences and restriction equations for given values of scaled variables. Nevertheless, for a bounded variable XY it is explicitly required that the main term of the confluence for ∂XY or the restriction equation for $\Box XY$ is accommodated to the boundary restriction $\triangleright XY$. For a given scale value of XY the value of $\triangleright XY$ is constant at $\{-,0\},\{-,0,+\}$, or $\{0,+\}$, but the general form of the confluence for ∂XY or the restriction equation for $\square XY$ must be described with the help of the symbol $\triangleright XY$, since the constant value depends in a specific way on the scale value of XY.

Before the conditions (c1) to (c10) can be stated, several definitions have to be introduced and something has to be said about the motivation of some of the conditions. A confluence always has a main term (see 2.4). This main term is an algebraic sum of constant and variable components. A **variable component** of the main term of a confluence is a current or lagged tendency with its sign in the algebraic sum. The constant direction sum is thought of as the combined effect of several constant influences. We refer to it as the **constant component**. Of course, a main term may have no constant components or no variable components, but it must have at least one component. The value of an empty main term is not defined.

A restriction equation also always has a main term (see 2.7). In the case of a restriction equation the **constant component** is not necessarily a direction sum but can be any convex direction set. The **variable components** of the main term of a restriction equation are not necessarily current or lagged tendencies with their signs in the algebraic sum but also boundary restrictions or system specific restrictions with their signs in the algebraic sum. Also the main term of a restriction equation may have no constant or no variable components, but it must have at least one component.

Some properties required by the conditions have the purpose to give a clear and simple structure to main terms. Unnecessary components are avoided. No component is permitted to appear more than once in the same main term. A convex direction set is not changed by adding it to itself. Therefore there is no need for multiple representation of the same component of the algebraic sum.

Zero is not permitted as the value of a constant component of a main term with at least two components. The main term has the same value whether zero is added or not. However, zero is not excluded as the value of the constant component if the main term has no variable components.

If the constant term of a main term is $\{-,0,+\}$ then it is not permitted to have any variable components. In this case variable components are superfluous, since they could not change the value $\{-,0,+\}$ of the main term.

A main term of a restriction equation with $\triangleright XY$ and $- \triangleright XY$ as variable components is not permissible. Since $\triangleright XY$ is either $\{-,0\}$ or $\{0,+\}$ or $\{-,0,+\}$ the sum $\triangleright XY - \triangleright XY$ can be replaced by $\{-,0,+\}$. However $\square XY$ and $-\square XY$ can occur in the same main term of a restriction equation. Since $\square XY = 0$ is not excluded the sum $\square XY - \square XY$ may have the values $\{-,0,+\}$ or zero.

We are now ready to state conditions (c1) to (c10). To some extent the conditions will repeat what has been said above, but there will also be additional requirements whose reasons will be explained later. In the conditions XY stands for an arbitrary variable.

(c1) **Form of confluences:** For given values of the scaled variables a confluence has one of the following three forms:
  (c 1.1) $\partial XY = T$
  (c 1.2) $\partial XY = T \ @ \ \rhd XY$
  (c 1.3) $\partial XY = T \ @ \ \Box XY$
  The confluence for ∂XY cannot have form (c 1.1) if XY is bounded and it cannot have form (c 1.2) if XY is unbounded.

(c2) **Form of restriction equations:** For given values of the scaled variables a restriction equation has one of the following two forms:
  (c 2.1) $\Box XY = S$
  (c 2.2) $\Box XY = S \ @ \ \rhd XY$
  The restriction equation for $\Box XY$ has the form (c 2.1) if XY is unbounded and form (c 2.2) if XY is bounded.

(c3) **Common structure of main terms:** A main term of a confluence or restriction equation is an algebraic sum with at least one component, at most one constant component, and finitely many variable components. Each component appears only once in the algebraic sum.

(c4) **Main terms of confluences:** A constant component of the main term of a confluence is a constant direction sum. A variable component of the main term of a confluence is a current or lagged tendency with its sign in the algebraic sum.

(c5) **Main terms of restriction equations:** A constant component of the main term of a restriction equation is a convex direction set. A variable component of the main term of a restriction equation is a current or lagged tendency, a boundary restriction or a system specific restriction, in all cases with its sign in the algebraic sum.

(c6) **Vanishing constant component:** A constant component of the main term of a confluence or restriction equation cannot have the value zero unless the main term has only one component.

(c7) **Full direction set as constant component:** If the constant component of the main term of a confluence or restriction equation has the value $\{-, 0, +\}$, then this main term does not have any variable components.

(c8) **Boundary restrictions in main terms of restriction equations:** The main term of a restriction equation (not necessarily for $\Box XY$) cannot have $\rhd XY$ and $- \rhd XY$ together as components.

(c9) **Exclusion of self-dependence in confluences:** The main term of the confluence for ∂XY does not contain ∂XY or $-\partial XY$ as one of its components.

(c10) **Exclusion of self-dependence in restriction equations:** The main term of a restriction equation for $\Box XY$ does not contain ∂XY or $-\partial XY$ and also not $\Box XY$ or $-\Box XY$ among its components.

Interpretation. The conditions (c1) to (c10) will be interpreted in the following. The structural properties required by (c1) and (c2) are based on the idea that boundary restrictions should be expressed where they apply, but not where they do not apply. Unlike boundary restrictions system specific restrictions can constrain tendencies of bounded and unbounded variables. A boundary restriction of a variable XY is absolutely binding. Therefore $\Box XY$ must be accommodated to it.

Condition (c3) requires that main terms are algebraic sums. This is natural for the main terms of confluences, which are interpreted as joint effects of the directions of individual influences. This is also the idea behind (c4). The situation is different for main terms of restriction equations. Here (c5) permits not only direction sums but also convex direction sets that are not necessarily direction sums. It is maybe useful to look at an example in order to explain the interpretation of an addition of such components. Let

$$T = \partial UV + \partial WZ$$

be the main term of the confluence for ∂XY and assume that XY is unscaled and that UV and WZ are scaled variables with the scales U, v for UV and W, z for WZ. If UV and WZ are at their top values then XY cannot be increased but otherwise all directions are possible. This is described by

$$\partial XY = (\partial UV + \partial WZ) \; @ \; \Box XY$$

and

$$\Box XY = \triangleright UV + \triangleright WZ$$

As long as at least one of both components $\triangleright UV$ and $\triangleright WZ$ has the value $\{-, 0, +\}$ the system specific restriction has the same value. At $UV = v$ and $WZ = z$ we have $\triangleright UV = \{-, 0\}$ and $\triangleright WZ = \{-, 0\}$ and therefore $\Box XY = \{-, 0\}$.

It is not always possible to model the main term of a restriction equation by the same algebraic sum for all combinations of values for scaled variables. This can be seen with the help of the following example. Let PD (production) and DE (demand) be unscaled variables and let

$$T = \partial DE$$

be the main term of the confluence for PD. Suppose that production needs labor input and capacity use in fixed proportion. Let LA with the scale B, f be the variable labor input and let CA with the scale B, c be the variable capacity use. (The symbols f, c, and B stand for "full employment", "capacity limit", and "below the upper limit"). Production is limited by each of the two variables LA and CA. It cannot increase if $LA = f$ or $CA = c$ holds. It can be seen without difficulty that an adequate system specific restriction for PD cannot be expressed by an algebraic sum of $\triangleright LA$ and $\triangleright CA$. Fortunately, this is not a problem for the theory proposed here. The limitation of PD by LA and CA is adequately expressed by

$$\partial PD = \partial DE \ @ \ \Box PD$$

and

$$\Box PD = \begin{cases} \{-, 0\} & \text{for} \quad LA = f \quad \text{or} \quad CA = c \\ \{-, 0, +\} & \text{else} \end{cases}$$

The permissibility of boundary restrictions in main terms of restriction equations is a modeling opportunity which does not prevent case distinctions concerning scale values.

Conditions (c6), (c7) and (c8) exclude redundancies in main terms. This has been explained before the statement of the conditions. It remains to comment on (c9) and (c10). It is not clear whether the exclusion of self-dependence in the sense of these conditions is really necessary for the derivation of the results of later chapters. One may even gain some formal advantages by abolishing it. However, a clear causal interpretation of a confluence or restriction equation seems to require the exclusion of self-dependence. This is important for a reconstruction of boundedly rational reasoning on economic dynamics.

Conditions (c9) and (c10) prevent circularities in the interpretation of single confluences and single restriction equations or of a confluence for ∂XY and a restriction equation $\Box XY$ taken together. Without (c9) and (c10) the interpretation could run into fixed point problems on this local level. Of course circularities cannot and should not be avoided on the global level of all confluences and restriction equations taken together. Circularities on the global level are an important driving force of qualitative dynamic models. Fixed point problems cannot be avoided on the global level. In fact, a state is the solution of a set of simultaneous confluences and restriction equations and therefore can be looked upon as a fixed point of the system. As we shall see in 2.9 it is not obvious that at least one state exists.

The interpretation of a system as a whole must be based on the analysis of the system. However, a single confluence or restriction equation has to be interpreted directly and in isolation before the beginning of analysis. Therefore it seems to be reasonable to avoid circularities on the local level.

Condition (c9) permits ∂XY^- and condition (c10) permits ∂XY^- and $\triangleright XY$ in the main term. This does not prevent clear causal interpretations on the local level. Lagged tendencies are taken from the past and boundary restrictions are determined by scale values. A scale value may be changed by a shift and a lagged tendency by a lag extinction. Shifts and lag extinctions cause transitions but during a transition lagged tendencies and boundary restrictions do not change. In this sense lagged tendencies and boundary restrictions causally precede current tendencies and system specific restrictions. During the transition current tendencies and system specific restrictions change in the course of the readjustment process which will be explained in chapter 4.

2.11. The anchoring requirement

The conditions (c1) to (c10) of 2.10 concern single confluences and restriction equations or the relationship between a confluence and a restriction equation connected to the same variable. The "anchoring requirement" is an additional condition imposed on the list Γ of confluences and restriction equations of a system base as a whole. Some further definitions are needed before this requirement can be expressed. These definitions are relative to a given system base $B = (\Lambda, \Gamma)$.

A **system piece** or shortly a **piece** appears on the right hand side of confluences and restriction equations and belongs to one of the following categories: 1) values of variables, 2) current tendencies, 3) lagged tendencies, 4) boundary restrictions, 5) system specific restrictions, 6) constant directions, 7) constant direction sets.

Now it will be explained what it means that a system piece is "anchored". This is done with the help of a recursive definition.

A: The following kinds of system pieces are **anchored**: values of scaled variables, lagged tendencies, boundary restrictions, constant directions and constant direction sets.

B: A current tendency is **anchored**, if all system pieces appearing on the right hand side of its confluence are anchored.

C: A system specific restriction is anchored if all system pieces on the right hand side of its restriction equation are anchored.

It is now possible to state the anchoring requirement:

Anchoring requirement:	All system specific restrictions are anchored.

The anchoring requirement facilitates the analysis. It is crucial for the derivation of results in later chapters. No other justification can be given. The anchoring requirement limits the scope of applications of the theory proposed here, but this limitation does not seem to be severe. Only system specific restrictions are required to be anchored.

It can be seen immediately that the anchoring requirement is satisfied for the models of Table 4 and Table 6. In both cases the only restriction equation is $\Box DE = \triangleright PD$. The applicability of the theory proposed here would be too much narrowed down if also current tendencies were required to be anchored. In the model of Table 4, the tendency ∂PD depends on ∂DE and ∂DE depends on ∂PD. Therefore neither ∂PD nor ∂DE are anchored. Business cycle models often involve similar circularities. In the modified model of Table 6, however, all current tendencies are anchored. This can be seen without difficulty.

Example (An example violating the anchoring requirement). Consider the following system base $B = (\Lambda, \Gamma)$: The list of variables contains two unscaled variables, XY and UV and no scaled ones. The list Γ of confluences and restriction equations is as follows:

$$\begin{aligned} \partial XY &= \{-\} \mathbin{@} \Box XY \\ \partial UV &= -\partial XY \\ \Box XY &= \{+\} + \partial UV \end{aligned}$$

It can be seen immediately that neither ∂XY nor ∂UV nor $\Box XY$ is anchored. The conditions (b1), (b2) and (b3) of 2.7 and the conditions (c1) to (c10) of 2.8 are satisfied. Let s be a state of (Λ, Γ). It will now be shown that no such state s exists. Assume $\partial XY = -$ at s. Then at s we have

$$\partial UV = + \text{ and } \quad \Box XY = +$$

This yields $\partial XY = +$ contrary to the assumption $\partial XY = -$. Assume $\partial XY = +$ at s. Then at s we have

$$\partial UV = - \text{ and } \quad \Box XY = \{-, 0, +\}$$

This yields $\partial XY = -$ contrary to the assumption $\partial XY = +$. Now assume $\partial XY = 0$ at s. Then at s we have

$$\partial UV = 0 \text{ and } \quad \Box XY = +$$

This yields $\partial XY = +$ contrary to the assumption $\partial XY = 0$. Since ∂XY must have one of the values $-, 0$, or $+$ at a state, it follows that a state for the base $B = (\Lambda, \Gamma)$ of this example does not exist.

Final definition of a system base: From now on a **system base** $B = (\Lambda, \Gamma)$ will always be a pair which satisfies conditions (b1), (b2) and (b3) of 2.7 and (c1) to (c10) of 2.10 as well as the anchoring requirement. These conditions are parts of the definition of a system base.

Every system base has at least one state. This will be the content of the corollary of theorem 1 in 4.6. The example examined above shows that the anchoring requirement is important in this respect.

2.12. Making hidden balances explicit

The tardiness assumption about restricted switches underlying Table 9 in 2.9 concerns restricted tendency switches from zero to $-$ or $+$. Consider a tendency ∂XY with a confluence of the form

$$\partial XY = T \ @ \ \Box XY$$

Suppose that at a state s we have $\partial XY = 0$ and $\Box XY = \{-, 0\}$. The switch of ∂XY from 0 to $-$ is tardy according to Table 9. It is implicitely assumed that the balance of the influences on the main term T is positive.

The specification of a state does not contain any information about the balance of influences on a main term with the value $\{-, 0, +\}$. In this sense this balance is **hidden**. It is often quite reasonable to assume that the balance is outside the restriction, but not always. Suppose for example that the restriction equation for $\Box XY$ is as follows

$$\Box XY = \begin{cases} \{-, 0, +\} & \text{for } VW = B \\ \{-, 0\} & \text{for } VW = B \end{cases}$$

where VW is a scaled variable with the scale b, B. Let s_- be a state with $VW = b$ and $\partial XY = 0$ from which the state s is reached by an immediate shift of VW from b to B. Moreover, assume that at s_- as well as at s the value of T is $\{-, 0, +\}$. In this situation the balance of influences on T should be zero at s since it was zero at $_-$. It is natural to proceed from this idea. Therefore the tardiness assumption about restricted switches is not adequate for the example under consideration.

The difficulty arises, since hidden balances of main terms are not specified by a state. However this can be changed by modeling the hidden balance as the tendency ∂BT of an unscaled variable BT. The confluence for ∂XY is replaced by the following confluences for ∂BT or ∂XY:

$$\begin{aligned} \partial BT &= T \\ \partial XY &= \partial BT \ @ \ \Box XY \end{aligned}$$

The confluences for tendencies other than ∂XY and all restriction equations remain unchanged. Thereby one receives a new base with essentially the same interpretation as the original one. We say that the new base results from the original one by **making the hidden balance of** T explicit.

In the new system the main term of the confluence for ∂XY is formed by the single tendency ∂BT and the value of this tendency is a part of the specification of the state. Therefore no hidden balance problem with respect to the main term of the confluence for ∂XY can arise in the new system.

It is clear that in principle all hidden balances can be made explicit. In this way every system can be transformed into a new one without hidden balances. Therefore hidden balances are a modeling problem rather than a substantial difficulty for the application of our theory.

Making hidden balances explicit increases the number of variables and thereby makes the analysis more complex. Therefore it is recommendable to rely on the tardiness assumption about restricted switches wherever there are no strong reasons against this.

Consider a confluence of the form

$$\partial XY = T \ @ \ \triangleright XY$$

Let XY be a scaled variable with a top point c. Let s be a state with $XY = c$. Moreover assume that at s the value of ∂XY is zero. Here it can be argued that the balance of influences on T must be positive, if one excludes the special case that XY has always been at its top point c in the past. The top point c cannot be reached from the range just below it unless ∂XY is positive there. This justifies the assumption that ∂XY is still positive when XY arrives at c. Therefore one can expect that it will rarely meet an application in which the hidden balance of a main term subject to a boundary restriction needs to be made explicit. However, this argument does not seem to be transferable to system specific restrictions.

CHAPTER 3

Transition causes and qualitative dynamic systems

3.1. Main transition causes

All definitions of this chapter refer to a fixed but arbitrary system base $B = (\Lambda, \Gamma)$ and the states for this base (see 2.7 and 2.11). The dependence on B will not always be expressed explicitly.

Up to now we have introduced three kinds of transition causes: Shifts, lag extinctions and tendency switches. We refer to these three kinds of transition causes as **main transition causes**. A fourth transition cause will be introduced in 3.3. In the following we recapitulate the definition of the three main transition causes.

A **shift** is the change of the value of a scaled variable to a neighboring one. An **upward** shift is **pending** at a state, if there the tendency of the variable is positive and its value is not the top value. Similarly, a downward shift is **pending** at a state, if the tendency is negative and the value is not the bottom value.

A **lag extinction** is the change of the value of a lagged tendency of a variable to the value of the current tendency of the variable. A lag extinction is **pending** at a state, if there the two values are different.

A **tendency switch**, or shortly a **switch**, is a movement of a current tendency ∂XY from a direction d_1 to a direction d_2 and is **pending** at a state s, if the following conditions 1) to 4) are satisfied:

1) The value of ∂XY at s is d_1
2) $d_2 \neq d_1$
3) d_2 is in the value of the right hand side of the confluence for ∂XY at s
4) If at s the value of the right hand side of the confluence for ∂XY is $\{-, 0, +\}$ then $d_2 \neq 0$

A tendency switch from d_1 to d_2 at s is **immediate** if the right hand side of the confluence has the value $\{-, 0, +\}$ and d_1 has the value zero at s. Otherwise it is **tardy**. The four conditions together with these definitions summarize what has been said in 2.9.

If a transition cause pending at a state becomes effective, then a readjustment process begins which finally leads to a new state. This readjustment process will be described in chapter 4. Values of variables and lagged tendencies are kept constant during the readjustment process whereas the values of current tendencies and of system specific restrictions may change.

Pieces which are listed in part A of the definition of being anchored (see 2.11) are referred to as **anchors**. Shifts and lag extinctions are changes of anchors. Accordingly these transition causes are called **reanchorings**. Tendency switches do not belong to the category of reanchorings. They do not involve a change of an anchor.

It already has been pointed out in 2.9 that there is an important difference between tendency switches and reanchorings. Shifts or lag extinctions always lead to a transition, once they become effective. Contrary to this a tendency switch may not be feasible. This will be the subject matter of the next section.

3.2. Feasibility of tendency switches and examples

3.2.1. Example. An example of a tendency switch is provided by state 4 of the simple business cycle model of Table 4. At this state we have

$$\partial IN = + \quad \text{and} \quad \partial PD = +$$

Moreover $\Box DE = \rhd PD$ equals $\{-, 0, +\}$. Therefore the right hand side of the confluence for ∂DE has the value

$$(\partial PD - \partial IN) \mathbin{@} \Box DE = \{-, 0, +\}$$

Consequently, a tardy tendency switch of ∂DE from $+$ to $-$ is pending at state 4. If the value of ∂DE is replaced by $-$, the confluence for ∂PD yields

$$\partial PD = \partial DE = -$$

This yields the heuristic conclusion that state 6 is reached by the switch of ∂DE from $+$ to $-$. As we shall see in chapter 4, the same result is obtained by the general procedure of the theory proposed here.

The transition from state 4 to state 6 has the economic interpretation that the upswing ends and a downswing begins before production reaches the capacity limit c. A tendency switch at state 8 results in an analogous stop of the downswing before b is reached. In the model of Table 4 the variable DE is the only one with the property that a tendency switch of this variable can be pending at a state. A tendency switch of ∂DE is not pending at a state unless the main term of the confluence for ∂DE has the value $\{-, 0, +\}$ there. This is the case if the two tendencies ∂PD and ∂IN have the same value $-$ or $+$. The states 4 and 8 are the only ones which satisfy this requirement.

3.2.2. The system A. Table 10 shows a system base with only one state. Even if this example is not a full fledged system we refer to it as "system A".

Figure 5 shows a graphical representation of this base. The graphical conventions used are the same ones as in Figure 1 with the only difference that the constant $\{-\}$ in the main term of the confluence for ∂AA is also represented by a rectangle.

Variables

AA, AB unscaled

Confluences

$\partial AA = \{-\} + \partial AB$

$\partial AB = -\partial AA$

States

state	∂AA	∂AB
1	$-$	$+$

TABLE 10. The system A

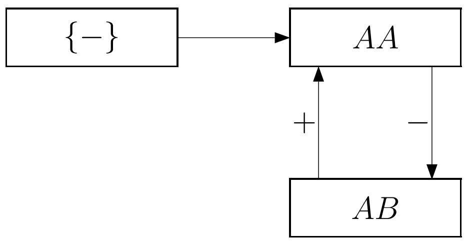


FIGURE 5. Graphical representation of the system A - An arrow indicates the influence of a tendency or a constant on the tendency of another variable. The arrow points to the variable with the influenced tendency. The sign at an arrow from a variable is the sign with which the influencing tendency appears in the main term of the influenced tendency. Constants enter the right hand side of a confluence as they are shown in their boxes. Therefore no sign is attached to an arrow from a constant to a variable.

It is clear that all confluences are satisfied at state 1. We now show that the system has no other states. This is done with the help of a case distinction with respect to the sign of ∂AB.

Case 1: $\partial AB = -$
In view of the confluence for ∂AB we must have $\partial AA = +$. However, for $\partial AB = -$ the right hand side of the confluence for ∂AA has the value $-$. This is a contradiction. Therefore the system A has no state with $\partial AB = -$.

Case 2: $\partial AB = 0$
In this case we must have $\partial AA = -$ and therefore $\partial AB = +$, contrary to the assumption $\partial AB = 0$. It follows that the system A has no state with $\partial AB = 0$.

Case 3: $\partial AB = +$
In view of the confluence for ∂AB we must have $\partial AA = -$ in this case, as in state 1. Consequently state 1 is the only one with $\partial AB = +$.

At state 1 the right hand side of the confluence for ∂AA has the value $\{-, 0, +\}$. Therefore a tendency switch from $-$ to $+$ is pending at state 1. However, since the system A has only one state, it does not permit any transition to another state with $\partial AB = +$. Even if the right hand side of the confluence for ∂AA has the value $\{-, 0, +\}$, the system as a whole enforces $\partial AA = -$. One must conclude that a tendency switch of ∂AA is pending but not feasible at the only state of system A.

At the moment the word "feasible" is used in an informal way. A more precise explanation of the term will be given later. One might think that a tendency switch is not really a transition cause, unless it is feasible. However the modeller may have to decide whether a tendency switch is plausible before the analysis shows whether it is feasible or not. We must think of a tendency switch as a hypothetical transition cause. The hypothesis that the transition is feasible may be confirmed or refuted by the analysis of the system. Nevertheless we continue to speak of tendency switches as transition causes regardless of whether they really cause transitions or not.

3.2.3. Feasibility, semifeasibility and hypothetical base. Reanchorings (see 3.1) do not share the hypothetical character of tendency switches. As we shall see in chapter 4 shifts and lag extinctions, once they become effective, always lead to a new state at which the shifted variable or the lagged tendency has the new value. This difference justifies a different treatment of reanchorings on the one hand and tendency switches on the other hand by the theory proposed here.

For the purpose of examining whether a switch of a tendency ∂XY from d_1 to d_2 at a state s for a base $B = (\Lambda, \Gamma)$ is feasible, a **hypothetical base** $B' = (\Lambda, \Gamma')$ for this switch will be used which is obtained from B as follows: The confluence for ∂XY is replaced by

$$\partial XY = d_2$$

and nothing else is changed.

It will now be shown that the hypothetical base B' is a base in the sense of the definition of 2.9. It is clear that the conditions (c1) to (c10) hold for B', but it remains to show that the anchoring requirement is satisified by B'. Obviously ∂XY is anchored in B' independently of which tendencies are anchored in B. Therefore every current tendency or system specific restriction which is anchored in B is also anchored in B'. Consequently B' satisifes the anchoring requirement.

The readjustment process in the hypothetical base B' leads to a new state s' for B'. This state s' may or may not be a state of the original base B. If s' is a state of B then the switch of ∂XY from d_1 to d_2 is **feasible at** s and s' is the new state reached in B. Otherwise the tendency switch is not feasible. The explanation of the term “feasible” is not yet complete since it will only be described in chapter 4 how a new state is reached by the readjustment process in the hypothetical base B'. Nevertheless the explanation given above is sufficient for the purposes of this chapter.

Consider the case that a tendency switch from $-$ to $+$ pending at a state s is not feasible. (The case of a switch from $+$ to $-$ is analogous.) In the example of system A any movement of ∂AA away from $-$ was impossible. However, this may be different in other systems. It is conceivable, that a system does permit a movement of ∂XY from $-$ in the direction of $+$, but this movement has to stop at $\partial XY = 0$. In the theory proposed here, a tendency switch from $-$ to $+$ may cause such a stopped movement. We refer to this possible consequence of a tendency switch from $-$ to $+$ as a **halfway switch** from $-$ to 0 at s.

Analogously, if a tendency switch from $+$ to $-$ pending at a state s is not feasible, a **halfway switch** from $+$ to 0 is a possible consequence of the switch from $+$ to $-$ at s.

If the hypothesis of a movement of ∂XY from $-$ to $+$ at s fails in spite of the fact that this tendency switch is pending at s, then the hypothesis of a halfway switch from $-$ to 0 has to be examined. This is done with the help of the **hypothetical base** for the halfway switch from $-$ to 0 at s. In this hypothetical base $B'' = (\Lambda, \Gamma'')$ the confluence for ∂XY is replaced by

$$\partial XY = 0$$

and everything else remains unchanged. In the same way as for the hypothetical base $B' = (\Lambda, \Gamma')$ for the tendency switch from $-$ to $+$ it can be seen that B'' is a base in the sense of 2.9. The hypothetical base for a halfway switch from $+$ to 0 is defined analogously.

Suppose that a tendency switch of ∂XY from $-$ to $+$ at a state s is not feasible. Let B'' be the hypothetical base for the halfway switch from $-$ to 0 at s. Then the switch of ∂XY from $-$ to $+$ at s is called **semifeasible**, if the new state s'' reached by the readjustment procedure in B'' is a state of $B = (\Lambda, \Gamma)$. Otherwise the switch from $-$ to $+$ is **infeasible**. The meaning of **semifeasible** and **infeasible** for a switch from $+$ to $-$ is analogous.

Of course, no halfway switch is possible for a tendency switch from d_1 to d_2 if either $d_1 = 0$ or $d_2 = 0$ holds. Such switches are **infeasible** if they are not feasible. No transition is caused by an infeasible halfway switch.

It is important to take notice of the fact that a hypothetical base for a tendency switch of ∂XY from d_1 to d_2 does not depend on the state s at which it is pending. If the same switch is pending at two states s_1 and s_2 then the same hypothetical base $B' = (\Lambda, \Gamma')$ is used for the examination of the feasibility of this switch at s_1 and s_2. However, the result of this examination may be different in the two cases. The readjustment processes used for this purpose run in the same hypothetical base, but they begin with different "starts". The concept of a start will be explained in 4.3. The start depends on the state, but not the hypothetical base. The same is true for the hypothetical base for a halfway switch of ∂XY for a switch of ∂XY from $-$ to $+$ or from $+$ to $-$.

We say that a halfway switch $\omega = [\partial XY \to 0]$ is **pending** at a state s if a tendency switch of ∂XY from $-$ to $+$ or from $+$ to $-$ is pending.

3.2.4. The system B. An example for a semifeasible tendency switch is provided by the system base shown by Table 11, referred to as "system B". Figure 6 graphically represents this base B. The figure makes use of arrows with hollow heads in order to indicate influences on restrictions.

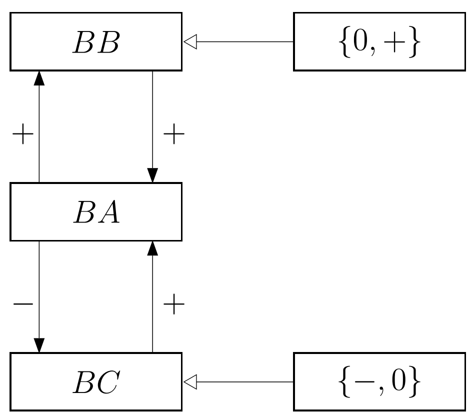


FIGURE 6. Graphical representation of the system B - An arrow with a hollow head indicates an influence on a restriction. A hollow headed arrow points to the variable in whose restriction equation the constant appears in the main term.

It can be seen easily that states 1 and 2 listed in Table 11 satisfy all confluences of system B. With the help of a case distinction with respect to the value of ∂BA we now show that there are no other states.

Case 1: $\partial BA = -$
In this case we have $\partial BB = \partial BC = 0$ in view of the restrictions by $\{0, +\}$ and $\{-, 0\}$ of ∂BB and ∂BC, respectively. This yields $\partial BA = 0$, contrary to the assumption $\partial BA = -$. Therefore system B has no states with $\partial BA = -$.

Case 2: $\partial BA = 0$
Obviously we have $\partial BB = \partial BC = 0$ in this case. Consequently state 2 is the only one with $\partial BA = 0$.

Case 3: $\partial BA = +$
Here we obtain $\partial BB = +$ and $\partial BC = -$. It follows that state 1 is the only one with $\partial BA = +$.

At state 1 the right hand side of the confluence for ∂BA has the value $\{-, 0, +\}$. Therefore a tendency switch from $+$ to $-$ is pending at state 1. Consider the hypothetical base obtained by replacing the main term of the confluence for ∂BA by $-$. In this hypothetical base ∂BA has the confluence

$$\partial BA = -$$

The other confluences of system B remain unchanged. It can be seen without difficulty that the hypothetical base has only one state. At this state we have $\partial BA = -$ and $\partial BB = \partial BC = 0$ and the right hand side of the confluence for ∂BA has the value zero. Consequently this state does not satisfy the original confluence for ∂BA. We can conclude that the tendency switch of ∂BA from $+$ to $-$ at state 1 is not feasible.

We now look at the hypothetical base for the halfway switch of ∂BA from $+$ to 0 at state 1. In this hypothetical base the original confluence for ∂BA is replaced by

$$\partial BA = 0$$

as the new confluence for ∂BA. The other confluences of system B remain unchanged. Obviously the hypothetical base for the halfway switch has only one state. At this state $\partial BA = \partial BB = \partial BC = 0$ holds. As we shall see in 4.10.5 the readjustment process in the second hypothetical base leads to this unique state. The original confluence for ∂BA is satisfied at this state. Therefore this state is also a state for the original base and it is the readjustment result, if the tendency switch of ∂BA from $+$ to $-$ at state 1 becomes effective. This switch is semifeasible.

Variables

BA, BB, BC unscaled

Confluences

$\partial BA = \partial BB + \partial BC$

$\partial BB = \partial BA \ @ \ \Box BB$

$\partial BC = -\partial BA \ @ \ \Box BC$

Restriction equations

$\Box BB = \{0, +\}$

$\Box BC = \{-, 0\}$

States

state	∂BA	∂BB	∂BC	$\Box BB$	$\Box BC$
1	+	+	−	$\{0, +\}$	$\{-, 0\}$
2	0	0	0	$\{0, +\}$	$\{-, 0\}$

TABLE 11. The system B

3.3. Perturbances, potential stationarity and auxiliary base

3.3.1. Concepts. Quantitative definitions of stability make use of small dislocations of stationary states. The underlying idea is that of a small exogenous disturbance of short duration. A stationary state is stable, if the system returns to it after such a perturbance. Otherwise it is unstable. In the theory proposed here, a similar question is asked about stationary states. However, in qualitative systems it is not obvious what is meant by a stationary state. The answer to this question must be deferred to the next section. The definition of stationarity depends on one part of the definition of a qualitative dynamic system which still needs to be introduced. However, it can be said already here that a stationary state must be "potentially stationary" in the sense of the following definition.

A state s is **potentially stationary**, if neither shifts nor lag extinctions nor immediate tendency switches are pending at s. This does not exclude the possibility that a tardy tendency switch is pending at a potentially stationary state. It may happen that a tardy tendency switch is pending at a potentially stationary state, but is excluded by assumption. This will become clear in Section 3.4.

Let $B = (\Lambda, \Gamma)$ be a base and let s be a potentially stationary state for B. Moreover let ∂XY be a current tendency for B and let d be a direction with

$d \neq 0$. The tendency ∂XY is called **perturbable by** d, if the following two conditions (i) and (ii) are satisfied

(i) At s the main term of the confluence for ∂XY has the value zero.

(ii) If the main term of the confluence for ∂XY is subject to a restriction $\triangleright XY$ or $\square XY$ then d is in the value of this restriction at s.

We say that a **positive perturbance of ∂XY is pending** at s if (i) and (ii) hold for $d = +$. Similarly, a **negative perturbance of ∂XY is pending at** s if (i) and (ii) hold for $d = -$.

A perturbance is thought of as a small temporary exogenous influence of short duration. A small positive or negative exogenous influence on ∂XY will not have any effect unless the value of ∂XY is zero at s as required by (i). If the value of ∂XY is $+$ then this value is not changed by a small negative influence. Similarly a small positive influence does not change the sign of ∂XY if this sign is $-$. However, if (i) is satisfied then even a small exogenous positive or negative influence will have an effect, unless this is prevented by a restriction $\triangleright XY$ or $\square XY$. Of course, if the restriction has the value $\{-, 0\}$ then a positive exogenous influence on the main term of the confluence for ∂XY cannot change the value of ∂XY to $+$. Therefore (ii) is required.

A perturbance of ∂XY is modelled as a change of the confluence for ∂XY. The exogenous influence d is added to the main term of this confluence. Thereby the original base B is changed to the **auxiliary base B_A for the perturbance of** ∂XY by d. This base $B_A = (\Lambda, \Gamma_A)$ differs from $B = (\Lambda, \Gamma)$ only by the confluence for ∂XY and by nothing else. Let T be the main term of the confluence for ∂XY in B and let T_A be the main term of the confluence for ∂XY in B_A. The main term T_A is not always the expression $T + d$, since this expression may not satisfy the conditions (c3), (c4), (c6), and (c7) required for main terms by 2.7. Equivalent transformations are applied to $T + d$ in order to obtain a simplified form which satisfies these conditions. Table 12 shows how this is done.

Six cases with respect to the structure of T are distinguished by Table 12. In case 1 the main term T has no constant component and $T + d$ satisfies (c3), (c4), (c6) and (c7). Therefore in this case is $T + d$. In the other 5 cases let C be the constant component of T. In these cases $T + d$ has two constants, C and d. The transformation **summation of constants** replaces $C + d$ by one constant C_0, the sum of C and d. This constant C_0 is d in case 2 in cases 2 and 3 and $\{-, 0, +\}$ in cases 4, 5, and 6. In cases 1, 2, 3, 4 and 6 the result satisfies (c3), (c4), (c6) and (c7) and is the main term T_A. In case 5 condition (c7) is not yet satisfied after the summation of constants. Here it is necessary to delete all variable components. This transformation is called **deletion of variable components.** A transformation is an **equivalent transformation** if it does not change the value

<table>
<tr><th colspan="6">Cases</th></tr>
<tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th></tr>
<tr><td rowspan="3">T has no constant component

$T_A = T + d$</td><td>$C = d$</td><td>$C = 0$</td><td>$C = -d$

T has no variable component</td><td>$C = -d$

T has at least one variable component</td><td>$C = \{-, 0, +\}$</td></tr>
<tr><td colspan="5">summation of constants</td></tr>
<tr><td>$T_A = T$</td><td>$T_A = d$</td><td>$T_A = \{-, 0, +\}$</td><td>deletion of variable components

$T_A = \{-, 0, +\}$</td><td>$T_A = \{-, 0, +\}$</td></tr>
</table>

TABLE 12. Simplification of $T + d$.*)

*) T is the main term of the confluence for ∂XY in B
C is the constant component of T, if there is any
T_A is the main term of the confluence for ∂XY in B_A

of the transformed expression for all possible combinations of values for its variable components. The deletion of variable components is an equivalent transformation, if it is applied to an expression with a constant component $\{-, 0, +\}$. It is clear that the summation of constants is always an equivalent transformation. After the deletion of variable components (c3), (c4), (c6) and (c7) are satisfied in case 5, too. At the bottom of Table 12 one finds the form of the main term T_A of the confluence for ∂XY in each of the 6 cases.

The main term T may depend on values of scaled variables. Different combinations of such values may give rise to different expressions for T and thereby also to different expressions for T_A. The positive perturbance of ∂GO in the model for Hume's specie flow mechanism provides an example (see 2.1 and 3.8.1). The confluence for ∂GO in the auxiliary base for this perturbance is as follows:

$$\partial GO = \begin{cases} \{-, 0, +\} & \text{for } TR = D \\ + & \text{for } TR = b \\ + & \text{for } TR = S \end{cases} \tag{1}$$

The consequences of the positive perturbance of ∂GO will be explored in 5.10.1.

It will now be shown that the auxiliary base B_A for a perturbance of ∂XY by d is always a base in the sense of the definition in 2.9. From what has been said above, it is clear that (c1) to (c10) always hold. It remains to be shown that B_A satisfies the anchoring requirement.

The confluences for tendencies other than ∂XY are the same ones in B and B_A. Therefore in all six cases of Table 12 the tendency ∂XY is anchored in B_A, if the pieces of T are anchored in B. Consequently every current tendency and every system specific restriction is anchored in B_A, if it is anchored in B. It follows that B_A satisifies the anchoring requirement and that B_A is a base in the sense of the definition in 2.9.

An auxiliary base B_A for a perturbance of ∂XY does not depend on the potentially stationary state at which it is pending. In this respect an auxiliary base is similar to a hypothetical one (see 3.2.3). If the same perturbance is pending at two different potentially stationary states s_1 and s_2 then the same auxiliary base is used for the examination of the consequences of the perturbance of ∂XY by d in both cases. However, here too, the “starts” are different for s_1 and s_2.

3.3.2. Interpretation and informal remarks. A perturbance of a tendency ∂XY modifies the main term of its confluence by the addition of an exogenous influence d. Thereby the original base B is changed to an auxiliary base B_A. Up to simplifying equivalent transformations the main term of the confluence for ∂XY in B_A is $T + d$.

The exogenous influence d on ∂XY is thought of as being of short duration. For a very short time the auxiliary base B_A replaces the original base B. The duration of the exogenous influence is not long enough to allow tardy transitions in the auxiliary base. However, any finite number of immediate transitions may take place in the auxiliary base. Immediate transitions are thought of as taking practically no time. Therefore the duration of the exogenous influence is not too short for a finite sequence of immediate transitions.

In the hypothetical base for a tendency switch or a halfway switch the right hand side of the confluence for the switched tendency is changed to a constant. Hypothetical bases can be defined in this simple way, since only one transition is explored in a hypothetical base and during this transition the values of scaled variables are constant. It does not matter how a hypothetical base is defined for other combinations of values of scaled variables. However, this is different for an auxiliary base. An immediate shift may change the combination of values of scaled variables. Therefore the main term of the confluence of the perturbed variable in the auxiliary base must correctly reflect the temporary exogenous influence at every state of this base which can be reached by immediate transitions. This is most easily achieved by a definition which covers all possible combinations of values of scaled variables.

3.3.3. Heuristic discussion of an example. We now turn our attention to the heuristic discussion of a specific example, namely, a positive perturbance of ∂DE at state 9 of the simple business cycle model shown by Table 4 in 2.5. Obviously state 9 is potentially stationary. It is also stationary in the sense of the definition which will be given in 3.4. The confluence for ∂DE does not depend on values of scaled variables. The main term of this confluence has no constant term. Therefore the confluence for ∂DE in the auxiliary base for the positive perturbance of ∂DE at state 9 is as follows:

$$\partial DE = (\partial PD - \partial IN + \{+\}) \mathbin{@} \Box DE$$

At state 9 we have $PD = n$ and therefore $\partial IN = 0$ and

$$\Box DE = \triangleright PD = \{-, 0, +\}.$$

The positive exogenous influence changes the value of the main term of the confluence for ∂DE at state 9 from 0 to $+$. Therefore one can expect that a state with $PD = n$ and $\partial DE = +$ will be reached in the auxiliary base. At such a state the confluence for ∂PD yields $\partial PD = +$. Moreover ∂IN as well as $\triangleright PD$ and $\Box DE$ have the same values as at state 9. We can heuristically conclude that this state is the first state of the auxiliary base reached from state 9 of the orginal base. (As we shall see in 4.10.3 the formal application of the theory proposed here yields the same result.) An immediate shift of PD from n to H is pending at this new state of the auxiliary system. As long as an immediate transition cause is pending at a state reached in the auxiliary system, the transition process stays in this system. The state with

$$PD = H \quad \text{and} \quad \partial IN = \partial PD = \partial DE = +$$

of the auxiliary system is the only one with $PD = H$ at which ∂PD and ∂DE have the same values as before. Therefore one can expect that this state is reached by the shift from n to H in the auxiliary system. (Here, too, the formal procedure of the theory proposed here comes to the same conclusion). No immediate transition cause is pending at this new state; moreover, it is also a state of the original system, namely state 4 of Table 5 in 2.5. With this state the sequence of transitions returns to the original system.

State 4 is a state of the cycle of the model of Table 4 (see Figure 3 in 2.5). The system does not return to the stationary state 9 from there. Therefore state 9 must be considered to be unstable with respect to a positive perturbance of ∂DE by any reasonable definition of stability. (According to the definition of stability given in 5.6, state 9 is unstable with respect to this perturbance.)

3.4. The four kinds of transition causes

Four kinds of transition causes have been described:

1. Shifts
2. Lag extinctions
3. Tendency switches
4. Perturbances.

A precise description of the notion of a qualitative dynamic system makes it necessary to introduce transition causes as formal objects. In order to make this clear a notation is adopted which denotes a transition cause by an expression in rectangular brackets:

$[XY \to V]$	immediate shift of XY to a range V
$[XY \to v]$	tardy shift of XY to a point v
$[\partial XY^-]$	lag extinction of ∂XY^-
$[\partial XY \to d]$	tendency switch of ∂XY to d
$[\partial XY : +]$	positive perturbance of ∂XY
$[\partial XY : -]$	negative perturbance of ∂XY

A transition due to a main transition cause (a shift, a lag extinction or a tendency switch) is called a **main transition**. Similarly, a transition due to an immediate transition cause is called an **immediate transition** and a transition due to a perturbance is called a **perturbance transition**.

In the case of a tendency switch $\omega = [\partial XY \to d]$ pending at a state s the readjustment process is applied in the hypothetical base $B_\omega = (\Lambda, \Gamma_\omega)$. As has been explained in 3.2 this base B_ω is obtained from $B = (\Lambda, \Gamma)$ by replacing the original confluence for ∂XY by $\partial XY = d$. Nothing else is changed. If it turns out that ω is not feasible at s, then the readjustment process is also applied to the halfway switch of ∂XY to zero at s in the hypothetical base for this halfway switch. We use the notation $\mu = [\partial XY \to 0]$ for the halfway switch associated to ω. However, it should be kept in mind that a halfway switch is not a transition cause, but a possibility which has to be explored, if a tardy tendency switch fails to be feasible. If ω is not feasible at s then the readjustment process is applied to the halfway switch in $B_\mu = (\Lambda, \Gamma_\mu)$. As has been explained in 3.2 this base B_μ is obtained from $B = (\Lambda, \Gamma)$ by replacing the original confluence for ∂XY by $\partial XY = 0$. Nothing else is changed.

3.5. The priority ranking

Usually several main transition causes are pending at a state. In such cases qualitative reasoning is often guided by plausibility judgments about which transition causes should be taken seriously and which ones should be neglected. Such judgments are assumptions rather than conclusions and therefore must be formalized as a part of a qualitative dynamic system.

Sometimes it is not sufficient to make a distinction between plausible and implausible transition causes. The system may have a cycle such that at every state of the cycle the same shift is pending. The shift may be implausible at every state of the cycle, but it must happen eventually. It therefore cannot be completely excluded from consideration. An example of this kind will be discussed in the next section. It may be necessary to form judgments about the order of priority in which transition causes at a state are considered. The notion of a priority ranking formally expresses such judgments.

The priority ranking concerns only main transition causes. Only at stationary states perturbances are considered. A stable stationary state is required to be stable against all plausible perturbances. Therefore it is unnecessary to distinguish degrees of plausibility as far as perturbances are concerned.

A **priority ranking** ρ for a system base (Λ, Γ) is a function which assigns a **rank** $\rho(\omega, s)$ **of** ω **at** s to every pair (ω, s) such that s is a state for (Λ, Γ) and ω is a main transition cause pending at s. The ranks $\rho(\omega, s)$ are non-negative integers. The **priority order** at a state s for (Λ, Γ) is the restriction of ρ to pairs (ω, s) with this s.

Rank 1 indicates the highest priority, rank 2 the second highest and so on. Rank zero means that the transition cause has no priority whatsoever and is simply omitted from consideration. Several main transition causes may have the same rank at the same state. The set of all transition causes with rank k at s is denoted by $\phi_k(s)$.

Priority rankings cannot be chosen completely arbitrarily. Some definitions need to be introduced before the statement of the conditions imposed on priority rankings.

A state s is called **fleeting** if at least one immediate transition cause is pending at s. A state s is **lasting**, if no immediate transition causes are pending at s. A **persistent** transition cause is a tardy shift or a lag extinction. An **exposed** state s is a lasting state at which at least one persistent transition cause is pending.

It can be seen without difficulty that a lasting state is potentially stationary if and only if it is not exposed. The following conditions (d1), (d2) and (d3) are imposed on the priority ranking ρ:

(d1) Only persistent transition causes can have ranks greater than 1. All other transition causes have ranks zero or 1.

(d2) At a fleeting state s the set $\phi_1(s)$ contains at least one immediate transition cause, but no tardy ones and $\phi_k(s)$ is empty for $k > 1$.

(d3) At an exposed state s the set $\phi_1(s)$ is non-empty and all persistent transition causes have positive ranks.

A state is either fleeting or exposed or potentially stationary. The three conditions do not explicitely mention potentially stationary states. However, no other main transition causes than tardy tendency switches can be pending at a potentially stationary state. By (d1) tardy tendency switches must have rank zero or 1. Therefore, the conditions (d1), (d2) and (d3) imply the following condition (d4):

(d4) At a potentially stationary state s the set $\phi_1(s)$ may or may not be empty and $\phi_k(s)$ is empty for $k > 1$.

The theory proposed here formalizes assumptions on the plausibility of main transition causes as a ranking rather than a set of plausible main transition causes. In this way difficulties can be overcome, which concern persistent transition causes at exposed states. Condition (d3) permits degrees of plausibility for persistent main transition causes at exposed states. For fleeting states and potentially stationary states $\phi_1(s)$ is a set of plausible main transition causes and $\phi_k(s)$ is empty for $k > 1$. This means that the concept of a priority ranking does not deviate more than necessary from that of a set of plausible main transition causes.

Consider an exposed state. Normally one would expect that a transition cause of rank 1 becomes effective at this state. However it may happen that as long as this is the case the system stays in a set of states with the property, that the same persistent transition cause of higher rank than 1 is pending at every state of this set. Even if this persistent transition cause is less plausible it must happen eventually. The modeller must adjust to the situation by enlarging the set of main transitions considered as plausible to persistent transition causes of higher rank. Condition (d3) prepares the ground for this.

It will be shown in chapter 5 that there cannot be an infinite sequence of immediate transitions. Therefore the difficulty of a persistent tardy transition cause pending at every state of such a sequence does not arise. It will also be shown in chapter 5 that immediate tendency switches are always feasible. Therefore it cannot happen, that only infeasible tendency switches are in $\phi_1(s)$ at a fleeting state s.

At an exposed state s the set $\phi_1(s)$ may contain only tardy tendency switches and all of them may be infeasible. In this case, too, an enlargement of the set of main transitions considered to be plausible beyond $\phi_1(s)$ is necessary. One could avoid this by the requirement that infeasible tardy tendency switches always must have rank zero. However, if this is done, one cannot specify the priority ranking before the analysis of the feasibility of tendency switches. It seems to be better to model initial plausibility expectations and a framework for adjusting them, if necessary.

We say that the priority order at a state s has a **gap** at rank j if $\phi_j(s)$ is empty but for some $k > j$ the set $\phi_k(s)$ is non-empty. It can be seen without difficulty that conditions (d1) to (d3) exclude a gap of rank 1. A gap at a greater rank j is possible but only at an exposed state. To some extent the theory proposed here makes use of rank comparisons among persistent transition causes at different exposed states. Therefore the possibility of ranks greater than 1 serves a useful purpose.

3.6. The perturbance assignment

In this section we shall explain how assumptions on the plausibility of perturbances are modelled by the theory proposed here. We begin with the definition of stationarity.

A potentially stationary state s has been defined as a state at which no other main transition causes than tardy tendency switches are pending (see 3.3). Moreover $\phi_k(s)$ is empty for $k > 1$ if s is potentially stationary (see (d4) in 3.5). A potentially stationary state s is **stationary**, if $\phi_1(s)$ is empty or contains no other transition causes than infeasible tardy tendency switches.

It is useful to distinguish two kinds of stationarity. A stationary state s is **ex ante stationary** if $\phi_1(s)$ is empty and **ex post stationary** otherwise. It can be seen before the determination of the feasibility of tardy tendency switches whether a state is ex ante stationary or not, but if it is not ex ante stationary it may still turn out that it is ex post stationary.

In the theory proposed here plausibility judgments on perturbances are formed for every potentially stationary state s for the case that it turns out to be ex post stationary if it is not ex ante stationary anyhow. A **perturbance assignment** α for a system base (Λ, Γ) is a function which assigns a set $\alpha(s)$ of perturbances at s to every potentially stationary state s for (Λ, Γ).

The definition of stability for a stationary state s will require stability against every perturbance $\omega \in \alpha(s)$. The set $\alpha(s)$ may be empty or non-empty. The plausibility judgments expressed by $\alpha(s)$ are conditional in the sense that they are thought of as reasonable in the case that s is stationary and irrelevant otherwise. We refer to $\alpha(s)$ as the **expected perturbance set** at s and to the elements of $\alpha(s)$ as **expected perturbances** at s.

3.7. The definition of a qualitative dynamic system

A **qualitative dynamic system**

$$\Phi = (\Lambda, \Gamma, \rho, \alpha)$$

consists of a system base (Λ, Γ), a priority ranking ρ for (Λ, Γ) and a perturbance assignment α for (Λ, Γ). It is maybe useful to recapitulate the definitions of the four parts of a qualitative dynamic system.

Λ is the **list of variables.** This list contains finitely many variables, scaled variables with their scales and unscaled variables (see 2.1).

Γ is **the list of confluences and restriction equations.** This list has the properties (b1), (b2) and (b3) of 2.7. Moreover the confluences and restriction equations in this list satisfy the conditions (c1) to (c10) of 2.8. In addition to this the anchoring requirement of 2.9 is satisfied for the list as a whole.

ρ is the **priority ranking.** The function ρ assigns a **rank** $\rho(\omega, s)$ to every pair (ω, s) such that s is a state of the system and ω is a main transition cause pending at s. The rank $\rho(\omega, s)$ is a non-negative integer. The priority ranking must satisfy the conditions (d1) to (d3) of 3.5.

α is the **perturbance assignment.** The function α assigns a set $\alpha(s)$ of expected perturbances at s to every potentially stationary state s.

All definitions in this chapter and chapters 4 to 7 will refer to a fixed but arbitrary qualitative dynamic system, unless they concern specific examples only. This will generally not be expressed explicitly.

It is possible that a qualitative dynamic system does not have any potentially stationary state. Of course, one does not have to specify a perturbance function if this happens. Formally, in this case α is an empty function which maps the empty set onto itself. Similarly it is not excluded that no transition causes are pending at any state. Then not only α but also ρ is the empty function.

3.8. Examples of qualitative dynamic systems

Up to now five examples of system bases have been described. In the following we shall specify a priority ranking and a perturbance assignment for each of these system bases. Thereby we obtain five examples of qualitative dynamic systems.

3.8.1. Hume's specie-flow mechanism. This model has been described in Sections 2.1 and 2.2. The only transition causes pending at states of this model are shifts and perturbances. Tardy shifts of TR to b are pending at states 1 and 3 and no other main transition causes. No main transition causes are pending at state 2. It follows by condition (d3) that the tardy shifts at states 1 and 3 must receive rank 1. Therefore the priority ranking is specified in this way, which is the only possible one.

The only potentially stationary state is state 2. In fact, this state is ex ante stationary, since no main transitions are pending there. We specifiy the perturbance assignment as follows: The expected perturbance set of state 2 has exactly two elements, namely the positive and negative perturbances of ∂GO. These perturbances are not the only ones pending at state 2. The tendencies of ∂DE, ∂EX, and ∂IM are also perturbable. However, the way in which the perturbance assignment is specified here, is akin to Hume's original argument.

3.8.2. The simple business cycle model of Table 4. In this case the priority ranking described by Table 13 suggests itself. The shifts pending at states 1 to 8 receive rank 1. These states belong to the cycle shown by Figure 3 in 2.6. All other main transition causes pending at states 1 to 8 receive rank zero. Thereby the tardy tendency switches pending at states 4 and 8 are excluded from consideration.

State 9 is the only potentially stationary state. No main transition causes are pending at this state. Therefore state 9 is ex ante stationary. The perturbance set for state 9 is specified as the set of the positive and negative perturbances of ∂IN.

state	PD	$\triangleright PD = \\ = \Box DE$	∂PD	∂DE	∂IN	priority rank 1*	expected perturbances
1	b	$\{0,+\}$	$+$	$+$	$-$	$[PD \to L]$	/
2	L	$\{-,0,+\}$	$+$	$+$	$-$	$[PD \to n]$	/
3	n	$\{-,0,+\}$	$+$	$+$	0	$[PD \to H]$	/
4	H	$\{-,0,+\}$	$+$	$+$	$+$	$[PD \to c]$	/
5	c	$\{-,0\}$	$-$	$-$	$+$	$[PD \to H]$	/
6	H	$\{-,0,+\}$	$-$	$-$	$+$	$[PD \to n]$	/
7	n	$\{-,0,+\}$	$-$	$-$	0	$[PD \to L]$	/
8	L	$\{-,0,+\}$	$-$	$-$	$-$	$[PD \to b]$	/
9	n	$\{-,0,+\}$	0	0	0	/**	$[\partial IN : +]$, $[\partial IN : -]$

TABLE 13. Priority ranking and perturbance assignment for the model of Table 4.

*All main transition causes which do not have rank 1, have rank zero
**No main transition causes are pending at state 9

One could also look at an alternative priority ranking for the model of Table 4. One could, for example, give rank 1 not only to the shifts pending at states 1 to 8 but also to the tardy tendency switches of ∂DE pending at states 4 and 8. In 3.2 the heuristic conclusion has been reached that a tendency switch of ∂DE at state 4 from + to − leads to state 6. The upswing ends and the downswing begins before the capacity limit c is reached. Essentially the same line of reasoning comes to the heuristic conclusion that a tendency switch of ∂DE at state 8 from − to + leads to state 2. The downswing ends and the upswing begins before the minimum production b is reached. The alternative priority ranking would result in a different picture of the cycle. The new picture leaves it open how the upswing and the downswing end. The movement of production may be reversed at the boundaries of its scale or before.

3.8.3. The modified simple business cycle model of Table 6. If a system has many states one may wish to specify the priority ranking and the perturbance assignment on the basis of general principles which can be applied to every state. Thereby one avoids the necessity of looking at every state separately. The priority ranking shown by Table 14 is based on the following principles applied to every priority order at a state:

1. All immediate transition causes have rank 1.
2. All tardy tendency switches have rank zero.
3. A lag extinction of ∂PD^- has priority over a tardy shift of PD at every exposed state at which both are pending
4. Priority orders have no gaps (see 3.5).

The perturbance assignment of Table 14 is based on the following principle:

5. At a potentially stationary state s all positive and negative perturbances of ∂IN are in $\alpha(s)$ but no other ones.

Together with the conditions (d1) to (d4) imposed on priority orders (see 3.5) these five principles fully determine the priority ranking and the perturbance assignment of Table 14.

state	PD	∂PD^-	∂PD	rank 1	rank 2	expected perturbances
1	b	−	0	$[\partial PD^-]$	/	/
2	b	−	+	$[PD \to L]$	/	/
3	b	0	+	$[PD \to L]$	/	/
						— *continuation next page*

Table 14: Priority ranking and perturbance assignment for the model of Table 6.

state	PD	∂PD^-	∂PD	rank 1	rank 2	expected perturbances
4	b	$+$	$+$	$[PD \to L]$	/	/
5	L	$-$	$-$	$[PD \to b]$	/	/
6	L	$-$	0	$[\partial DE \to -]$, $[\partial DE \to +]$	/	/
7	L	$-$	$+$	$[\partial PD^-]$	$[PD \to n]$	/
8	L	0	$+$	$[\partial PD^-]$	$[PD \to n]$	/
9	L	$+$	$+$	$[PD \to n]$	/	/
10	n	$-$	$-$	$[PD \to L]$	/	/
11	n	0	0	/	/	$[\partial IN : -]$, $[\partial IN : +]$
12	n	$+$	$+$	$[PD \to H]$	/	/
13	H	$-$	$-$	$[PD \to n]$	/	/
14	H	0	$-$	$[\partial PD^-]$	$[PD \to n]$	/
15	H	$+$	$-$	$[\partial PD^-]$	$[PD \to n]$	/
16	H	$+$	0	$[\partial DE \to -]$, $[\partial DE \to +]$	/	/
17	H	$+$	$+$	$[PD \to c]$	/	/
18	c	$-$	$-$	$[PD \to H]$	/	/
19	c	0	$-$	$[PD \to H]$	/	/
20	c	$+$	$-$	$[PD \to H]$	/	/
21	c	$+$	0	$[\partial PD^-]$	/	/

Table 14: Priority ranking and perturbance assignment for the model of Table 6.

The example shows that one does not need a complete overview over all possible states, before a priority ranking and a perturbance assignment can be specified. Of course, different principles may be adequate for different models.

3.8.4. The system A. This system has only one state (see 3.2.2). The priority ranking and the perturbance assignment of system A are shown by Table 15.

state	∂AA	∂AB	priority rank 1	expected perturbances
1	$-$	$+$	$[\partial AA \to +]$	/

TABLE 15. Priority ranking and perturbance assignment of system A.

Only one transition cause is pending at the only state, the tendency switch of ∂AA which receives the priority rank 1. State 1 is potentially stationary and in fact ex post stationary since the tendency switch is infeasible (see 3.2.2). As no perturbances are pending at state 1, the expected perturbance set of this state is necessarily empty.

Of course, it makes no sense to assign rank 1 to a tendency switch at state 1, once it is known that the system has no other states. However the ranking may be derived from a general principle, e.g., that every main transition cause pending at an exposed state receives rank 1.

3.8.5. The system B. The system has two states 1 and 2 (see 3.2.4). The priority ranking and the perturbance assignment are shown by Table 16. At state

state	∂BA	∂BB	∂BC	$\Box BB$	$\Box BC$	priority rank 1	expected perturbances
1	$+$	$+$	$-$	$\{0,+\}$	$\{-,0\}$	$[\partial BA \to -]$	/
2	0	0	0	$\{0,+\}$	$\{-,0\}$	/	$[\partial BA : -]$, $[\partial BA : +]$

TABLE 16. Priority ranking and perturbance assignment of system B.

1 only one main transition cause is pending, the tendency switch of ∂BA from $+$ to $-$, which receives the priority rank 1. No main transition cause is pending at state 2. Each of the two states is potentially stationary. However no tendencies are perturbable at state 1. Therefore the expected perturbance set of state 1 is empty. At state 2 all tendencies are perturbable but only the negative and positive perturbances of ∂BA are in the expected perturbance set specified by Table 16.

3.8.6. Further remarks. It can be seen without difficulty that the four systems described in 3.8 satisfy the conditions (d1), (d2), and (d3) imposed on priority rankings in 3.5. The notion of a priority ranking by general principles is intentionally not made precise. A precise definition would require an exact description of the properties of a state to which such principles can refer. This would unnecessarily restrict the space of possible priority rankings. There may be applications which require deviations from general principles for particular states.

Therefore it seems to be preferable to preserve the flexibility gained by permitting any specification of the priority ranking in agreement with (d1), (d2), and (d3).

3.9. The system C

It may be hard to understand why priority rankings need to be specified and not just sets of transition causes which should be taken into account. In the following this will be explained with the help of an example. This example is the "system C" described by Table 17.

Variables

CA	scale B, c
CB, CC	unscaled

Confluences

$\partial CA = \{+\}$ @ $\triangleright CA$

$\partial CB = \partial CA + \partial CC$

$\partial CC = \{-\}$

States and priority ranking

					priority order	
state	CA	∂CA	∂CB	∂CC	rank 1	rank 2
1	B	$+$	$-$	$-$	$[\partial CB \to +]$	$[CA \to c]$
2	B	$+$	0	$-$	$[\partial CB \to -], [\partial CB \to +]$	/
3	B	$+$	$+$	$-$	$[\partial CB \to -]$	$[CA \to c]$
4	c	0	$-$	$-$	/	/

Perturbance assignment

The expected perturbance set for the potentially stationary state 4 is empty.

TABLE 17. The system C

System C has only 4 states. No transition causes are pending at state 4. Therefore no priority order is assigned to state 4. The state 4 is potentially stationary and in fact ex ante stationary. The expected perturbance set for state 4 must be empty since no perturbances are pending at this state. The states 1 and 3 are exposed, since the tardy shift $[CA \to c]$ is pending at these states. This shift receives rank 2 at states 1 and 3 and the tardy tendency switches pending at these states have rank 1. State 2 is fleeting. The immediate tendency switches at state 2 have rank 1 and the shift $[CA \to c]$ pending at state 2 receives rank

0 at this state as required by (d2). It can be seen immediately that the priority ranking satisfies conditions (d1), (d2), and (d3).

In chapter 4 it will become clear that all the tendency switches of ∂CB pending at states 1, 2, and 3 are feasible and lead to the new state obtained if the value d_1 of ∂CB is replaced by the value d_2 to which ∂CB is switched. No further adjustment is needed in all these cases. Similarly the shift $[CB \to c]$ always leads to state 4, the only state with $CB = c$.

Figure 7 shows the transition diagram for system C.

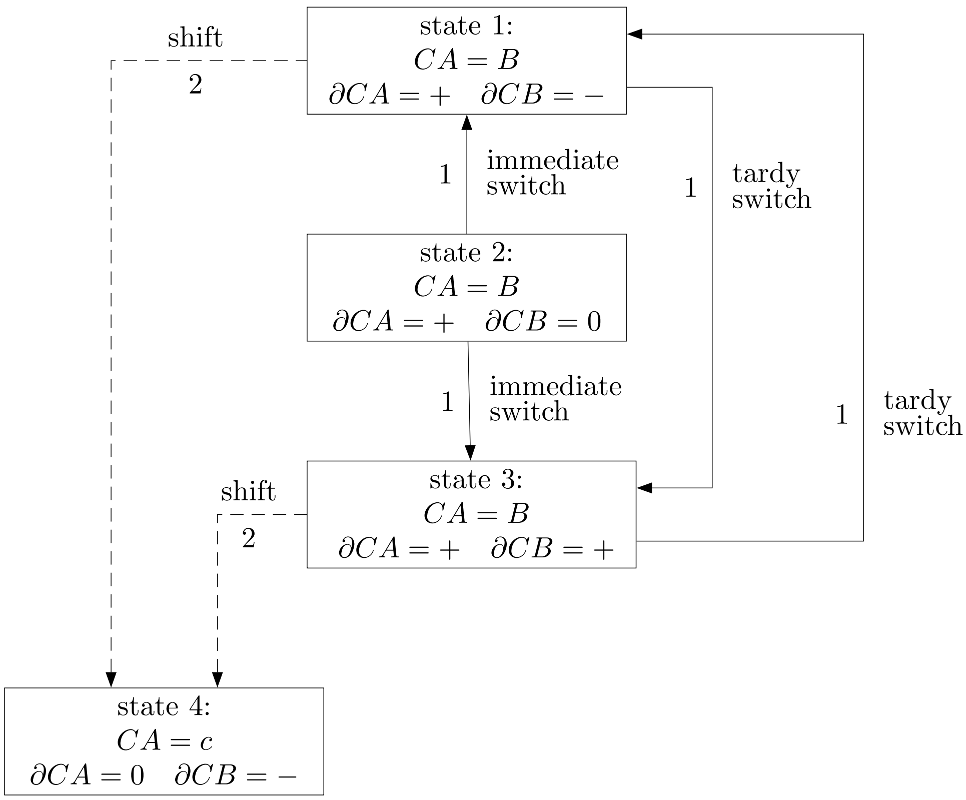


FIGURE 7. The transition diagram for system C

The transitions due to causes of highest priority are shown by unbroken lines and those of rank 2 by broken lines. The rank of a transition is also indicated as a number at the corresponding lines.

Suppose that in the system C only the transition causes of highest rank are considered. This means that a transition starting from states $1, 2$, or 3 always leads to one of these three states. A transition due to an immediate tendency switch at state 2 moves the system to state 1 or state 3. Once one of these two states has been reached, the system alternates between them. State 4 can never be reached. However, this is not compatible with the heuristic principle that a shift pending for a long time must happen eventually (see 2.6). Therefore one cannot restrict oneself to the specification of the transition causes of the highest priority. Transitions of lower priority may have to be taken into account in the construction of a transition diagram. In the next section it will be explained how the priority ranking enters the construction of the transition diagram.

3.10. Transition diagrams and permissible paths

All definitions of this section and the following chapters are relative to a fixed but arbitrary system $(\Lambda, \Gamma, \rho, \alpha)$. As has been explained at the end of 3.7 a system is formed by a base (see 2.7 and 2.9) together with a priority ranking ρ and a perturbance assignment α for this base.

Formally a transition diagram is a valued directed graph with the additional feature that a transition cause is associated to each edge. The vertices stand for the states and the edges show possible transitions. Each edge has a positive integer value, the rank of the transition cause associated to it.

Transition diagrams show only main transitions. The **rank of a transition** is the rank of the underlying transition cause. The possible consequences of perturbances are included in extended transition diagrams which are not explained here, but later in Section 4.4.

The **tentative transition diagram** shows all states and all main transitions of positive rank. The name “tentative” is attached to this diagram since it is not yet the transition diagram which will be defined later. The transition diagram will show only transitions up to a certain rank. As in the case of system C one may have to include transitions of lower priority but one does not have to go further than necessary in this respect.

The construction of the tentative transition diagram involves the readjustment process which will be introduced in chapter 4. For any given pair (ω, s) such that ω is a main transition cause pending at a state s the readjustment process uniquely determines a new state s', the result of the transition caused by ω at s. In the following we shall not be concerned with the construction of the tentative transition diagram but rather with the way in which the priority ranking enters the derivation of the transition diagram from the tentative transition diagram. For this purpose we can look at the tentative transition diagram as given.

A **tentative path** is a finite or infinite sequence of states continued as long as possible in such a way, that a main transition of positive rank leads from one state to the next. In the case of a finite tentative path the last state reached must be a state at which no main transition cause of positive rank is pending. If a tentative path begins at such a state it also ends there. This degenerate case is not excluded. A transition **on** a tentative path is a transition from one state on the path to the next one. The **rank** of a tentative path is the maximum of all ranks of transitions on the path, if there is at least one transition on the tentative path. Otherwise the **rank** of the path is 1.

A state may be reached more than once on a tentative path. Therefore one speaks of the m-th member of the sequence as the **m-th episode** of the path. In this way one can refer to a state together with its place in the path.

We say that a tentative path has an **unresolved shift**, if from some episode on, the same shift is pending at this episode and all later ones. Here the words "the same shift" mean that the variable and its values from which and to which the shift proceeds are always the same. Clearly, a reasonable path should not have an unresolved shift.

Not only shifts but also lag extinctions may be unresolved. In order to describe what is meant by this it is convenient to introduce the following manner of speaking: $+$ is **above** zero and $-$, and zero is **above** $-$. Similarly $-$ is **below** zero and $+$, and zero is **below** $+$. A tentative path has an **unresolved lag extinction**, if from some episode on the value of a lagged tendency ∂XY^- does not change and stays always above or always below the corresponding current tendency ∂XY. A tentative path is a **permissible path**, if it neither has an unresolved shift nor an unresolved lag extinction.

As the example of system C shows, the absence of unresolved shifts and lag extinctions is a highly desirable feature of a tentative path. A permissible path of rank 1 starting with a given state does not always exist. This happens for states $1, 2$, and 3 of system C. In such cases one has to look for a permissible path of higher rank. The concept of a priority ranking makes this possible.

Condition (d3) of 3.5 imposed on priority rankings requires that tardy shifts and lag extinctions have positive ranks at lasting states. This prevents the possibility that a shift or a lag extinction remains unresolved simply because it receives rank zero wherever it is pending. Therefore condition (d3) of 3.5 is imposed on priority rankings. At the moment we cannot yet prove that a permissible path starting with a given state always exists. This will be done in 5.7 on the basis of properties of the readjustment process which has not yet been defined. Therefore the definition of a transition diagram given below avoids this question.

We say that the tentative transition diagram is **ill structured**, if for at least one state a permissible path starting with this state does not exist. Otherwise the tentative transition diagram is called **well structured**.

We now define the **rank** of a well structured tentative transition diagram. This rank is the lowest integer k^* such that for every state s a permissible path starting at s with a rank $k \leq k^*$ can be found. The **transition diagram** derived from a well structured tentative transition diagram shows all main transitions of ranks $1, ..., k^*$ and no others. In other words, the transition diagram results from the tentative transition diagram by the deletion of all transitions of ranks greater than k^*.

A path **in** the transition diagram is a tentative path whose rank k is at most k^*. Obviously a permissible path starting with a given state can always be found among the tentative paths in a transition diagram which is derived from a well structured tentative transition diagram.

CHAPTER 4

Readjustment

4.1. Prestates

A transition from one state to another begins with a transition cause becoming effective. An initial local change of a scaled variable, a lagged tendency or a confluence has repercussions throughout the system. Confluences and restriction equations are upset and have to be readjusted. Such readjustments may disturb other confluences and restriction equations and their adjustment may have further repercussions and so on.

It is a core problem of the theory proposed here, to model a reasonable readjustment process with good mathematical properties. Starting from any state and any transition cause pending there, the readjustment process should converge in finitely many steps to a new state.

The readjustment process can be thought of as the description of a quick dynamics. States are **balanced** in the sense that all confluences and restriction equations are satisfied. This balance is absent in the quick dynamics. Nevertheless it must run in some space. The points of this space are the "prestates" which will be described in the following.

A state is a specification of values for all scaled variables, all current and lagged tendencies, and all system specific restrictions, such that all confluences and restriction equations are satisfied. A prestate is similar but with some important differences. One of these differences is a double representation of each current tendency ∂XY by a left tendency ∂XY_L and a right tendency ∂XY_R.

The **left** tendency ∂XY_L is the value of ∂XY on the left hand side of its confluence and the **right** tendency ∂XY_R is the value of ∂XY on the right hand sides of confluences and restriction equations. The left and right tendencies of a variable may be temporarily different during the readjustment process.

The left tendency ∂XY_L represents the current value of ∂XY whereas ∂XY_R represents its influence on other tendencies and system specific restrictions. At a prestate we may have $\partial XY_L = +$ or $\partial XY_L = -$ and at the same time $\partial XY_R = 0$. This means that the value of ∂XY is unequal to zero, but its influence is so weak in comparison to other tendencies that it is adequately represented by $\partial XY_R = 0$. In fact this is the only way in which ∂XY_L and ∂XY_R can be different during the readjustment process. In the course of this process we always have $\partial XY_R = 0$ for $\partial XY_L \neq \partial XY_R$.

It is convenient to have a common name for the components of the new state which are determined by the readjustment process. The notion of a directional serves this purpose. A **directional** is either a current tendency or a system specific restriction.

In addition to values of scaled variables, lagged tendencies, left and right current tendencies, and system specific restrictions a prestate specifies a **confirmation status** for every directional. The confirmation status of a directional is either L (**loose**) or F (**firm**). Accordingly we speak of loose and firm directionals.

A firm directional has already found its final value and is not changed any more by the readjustment process. At the beginning all directionals are loose but during the process more and more of them become firm. Whether a directional is changed or not depends on what is firm on the right hand side of its confluence or restriction equation. Therefore it is necessary to keep track of the confirmation status. We are now ready for the definition of a prestate.

A **prestate** is a specification of values for all scaled variables, for all lagged tendencies, for the left and the right tendencies of all variables, for all system specific restrictions, and for the confirmation status of each directional. The value of a scaled variable is on its scale, the values of lagged and of left and right tendencies are directions, the values of system specific restrictions are convex direction sets and the value of the confirmation status of a directional is either L or F.

A prestate p can be thought of as a vector with components for every item specified. In this sense we speak of the **components** of a prestate. We now explain what it means that a confluence, or restriction equation is satisfied at a prestate p. The **value of the right hand side** of a confluence or restriction equation at p is obtained by inserting the values of the corresponding right tendencies for current tendencies and, of course, the values of other components of p.

A confluence for a tendency ∂XY is **satisfied** at a prestate p, if at p the value of ∂XY_L is in the value of the right hand side of the confluence for ∂XY. A restriction equation is **satisfied** at a prestate p if at p the value of the left hand side of this restriction equation is equal to the value of its right hand side.

Note that a confluence for a tendency ∂XY can be satisfied at a prestate p with $\partial XY_L \neq \partial XY_R$. In fact, it does not even depend on ∂XY_R whether the confluence for ∂XY is satisfied or not, since a current tendency does not appear on the right hand side of its own confluence.

4.2. Operations

A directional is **adjusted** at a prestate p, if there its confluence or restriction equation is satisfied. Otherwise it is **maladjusted.** A directional is called **mature** at a prestate p, if the value of the right hand side of its confluence or restriction equation is fully determined by firm directionals. This does not exclude the possibility that some of the directionals appearing on the right hand side are loose; but whatever their value will be at the end, the value of the right hand side cannot change any more in the course of the readjustment procedure.

Consider the example of a confluence

$$\partial XY = T \mathbin{@} \Box XY$$

Suppose that at a prestate p we have $\Box XY = +$ and $\Box XY$ is firm. Then ∂XY is mature, regardless of whether loose tendencies appear on the right hand side or not.

A directional which is not mature is called **immature**. A **split** tendency at p is a current tendency ∂XY with $\partial XY_L \neq \partial XY_R$ at p: If we have $\partial XY_L = \partial XY_R$ at p, then ∂XY is called **univalued** at p. A current tendency ∂XY is a **non-zero** tendency at p if there $\partial XY_L = -$ or $\partial XY_L = +$ holds. If $\partial XY_L = 0$ holds at p then ∂XY is called a **zero tendency** at p. The distinction between non-zero and zero tendencies at a prestate p concerns left tendencies only.

The readjustment process can be looked upon as a procedure for the determination of the next state in a transition. An application of this procedure leads to a sequence of prestates which is continued as long as there are loose directionals. The steps from one prestate to the next involve three operations to be described in the following. The way in which these operations enter the process will be explained in Section 4.5.

Adaptation: Let t be the value of ∂XY_L and W be the value of the right hand side of the confluence for ∂XY at a prestate p. Adaptation of ∂XY at p means that the value of ∂XY_L becomes $t' = t \mathbin{@} W$ and nothing else is changed. This yields a new prestate p' which **results from** p **by the adaptation of** ∂XY.

Adaptation of a system specific restriction $\Box XY$ at p means that the value of $\Box XY$ specified by p is replaced by the value of the right hand side of the restriction equation for $\Box XY$ at p. Nothing else is changed. This yields a new prestate p' which **results from** p **by the adaptation of** $\Box XY$.

The adaptation of a directional at a prestate p does not lead to a different prestate, unless the directional is maladjusted at p. Nevertheless it is convenient to define the adaptation operation in a way which permits its application to adjusted directionals.

Dampening: This operation is only applied to immature univalued maladjusted non-zero tendencies ∂XY. At a prestate p let ∂XY be such a tendency. Dampening means that ∂XY_R is changed to zero. Nothing else is changed. This yields a new prestate p' which **results from** p **by dampening of** ∂XY.

Confirmation: This operation is only applied to loose adjusted directionals. Confirmation of a loose adjusted directional at a prestate p always changes the confirmation status of the directional from L to F. Only in the case of a loose adjusted split tendency ∂XY something else happens in addition to this. The value of ∂XY_R is changed to the value of ∂XY_L. Nothing else is changed. This yields a new prestate p', which **results from** p **by the confirmation of** the directional.

It is maybe useful to make some remarks about the interpretation of the three operations. Consider a maladjusted tendency ∂XY at a prestate p. Obviously at p the value t of ∂XY_L is not in the value W of the right hand side of the confluence for ∂XY. The change to the new value $t' = t$ @ W is the smallest one which achieves adjustment. The interpretation of the adaptation operation in the case of a system specific restriction is straightforward.

The operation of dampening removes the influence of a maladjusted univalued non-zero tendency ∂XY on other directionals. ∂XY_R is changed to zero but ∂XY_L remains unchanged. In terms of an underlying quantity this can be interpreted as follows. The time derivative of the quantity decreases in absolute value without changing its sign. Thereby the influence of this quantity on other variables becomes insignificant.

Consider the following specific example. Suppose that at the prestate p we have $\partial XY_L = \partial XY_R = +$ but the right hand side of the confluence for ∂XY has the value $-$. Assume that the right hand side keeps the value $-$ up to the end of the readjustment process. This means that eventually ∂XY_L and ∂XY_R have to change their value to $-$. What does this mean in terms of the time derivative of an underlying quantity? This time derivative decreases and must pass the value 0 before it becomes negative. Before it reaches 0 it becomes so small that its influence on other variables becomes practically zero. This is mirrored by the operation of dampening. Later the time derivative becomes negative, but at first it remains small in absolute value. This is captured by an adaptation of ∂XY which changes ∂XY_L to $-$ but leaves ∂XY_R at zero. As the time derivative continues to decrease, it regains its influence on other variables. This is reflected by the confirmation operation which finally changes the value of ∂XY_R to $-$.

The example shows why it is reasonable to apply the operations of dampening, adaptation and confirmation in this order and to separate them from each other.

Admittedly the interpretation in terms of an underlying quantity is by no means rigorous. It also does not really guide our definition of the readjustment process. As we shall see mature tendencies are adapted and confirmed without dampening. The main reason for applying dampening to immature univalued maladjusted non-zero tendencies is the removal of their influence on other still undampened maladjusted non-zero tendencies. However, the necessity of doing this cannot be explained before the definition of the readjustment process has been given.

4.3. Starts

Let s be a state of a qualitative dynamic system $\Phi = (\Lambda, \Gamma, \rho, \alpha)$ and let ω be a transition cause pending at s. The readjustment process begins with a prestate $p_0(\omega, s)$ called the **transition start for** ω **at** s. In the following we first define a prestate $p_0(s)$, the **prestate of** s, and then explain how the transition start $p_0(\omega, s)$ differs from $p_0(s)$.

The prestate $p_0(s)$ specifies the values of all scaled variables, lagged tendencies, and system specific restrictions in the same way as s. For every current tendency ∂XY the left tendency ∂XY_L and the right tendency ∂XY_R in $p_0(s)$ have the same value as ∂XY in s. Moreover all directionals have the confirmation status L.

We now must make a case distinction according to the nature of the transition cause.

Shifts: $\omega = [XY \to v]$ or $\omega = [XY \to V]$
In this case the value of XY in $p_0(s)$ is changed to v or V, respectively. Nothing else is changed. This yields $p_0(\omega, s)$.

Lag extinction: $\omega = [\partial XY^-]$
The value of ∂XY^- in $p_0(s)$ is replaced by the value of ∂XY in $p_0(s)$. Nothing else is changed. This yields $p_0(\omega, s)$.

Tendency switch: $\omega = [\partial XY \to d]$
Here we have $p_0(\omega, s) = p_0(s)$. If a tardy tendency switch ω is not feasible then also the halfway switch $\mu = [\partial XY \to 0]$ needs to be examined (see 3.2). The prestate $p_0(s)$ is also the transition start $p_0(\mu, s)$ for the halfway switch.

Perturbance: $\omega = [\partial XY : d]$
Here, too, we have $p_0(\omega, s) = p_0(s)$.

In the case of a reanchoring or in other words, a shift or a lag extinction, the readjustment process starting with $p_0(\omega, s)$ is run in the original system Φ. This means that operations applied to ∂XY in a step of the process make use of the confluence for ∂XY in Φ.

The other two transition causes involve changes of the confluence for ∂XY. Thereby the base B of Φ is changed to a modified structure. In the case of a tendency switch $\omega = [\partial XY \to d]$ this modified structure is the hypothetical base B_ω for ω or, after a tardy tendency switch ω has turned out not to be feasible, the hypothetical base B_μ for the halfway switch $\mu = [\partial XY \to 0]$. The readjustment process starts with $p_0(s)$ in B_ω as well as in B_μ.

In the case of a perturbance $\omega = [\partial XY : d]$ the modified base is the auxiliary base for ω (see 3.3). The readjustment process starting with $p_0(s)$ in the auxiliary base B_ω leads to a new state a_0 of B_ω. We call a_0 the **opening state** of the auxiliary base B_ω. From there on one may have to examine **immediate transition chains** $a_0, a_1, \ldots, a_M$ of states of B_ω with the property that for $m = 1, \ldots, M$ the state a_m is reached by an immediate transition from a_{m-1} to a_m in B_ω. An immediate transition chain is continued until a lasting state a_M for Φ_ω is reached. (In chapter 5 it will be shown that this must happen eventually.) From there on the system Φ is reentered.

The reentry happens at a prestate $p_0(a_M)$ called **return start**. The return start is defined in the same way as $p_0(s)$ with a_M instead of s. More about the consequences of perturbances will be said in 5.8. The base B of Φ and the auxiliary base have the same list of variables Λ, but the system of confluences and restriction equations is different in the two basees. Therefore a state for the auxiliary base is not necessarily a state for the original system. However, B and B_ω have the same space of prestates. Therefore a readjustment process in B_ω can start with $p_0(s)$ and a readjustment process in B can begin with a return start $p_0(a_M)$.

Return starts and the four kinds of transition starts have something in common, captured by the following definition: A **start** is a prestate p_0 with $\partial XY_L = \partial XY_R$ for every current tendency ∂XY and with the property that at p_0 all directionals are loose. It is clear that all transition starts and return starts are starts.

The readjustment process always begins with a start p_0. It does not matter for many properties of the readjustment process what kind of start this is.

It will be shown that a readjustment process beginning with a start p_0 always leads to a final prestate p' at which $\partial XY_L = \partial XY_R$ holds for all tendencies and all directionals are adjusted and firm. Such prestates are called **saturated**. A saturated prestate p' **generates** a new state s'. This state s' specifies the values of all scaled variables, lagged tendencies and system specific restrictions in the same way as p' and in s' each tendency ∂XY has the common value of $\partial XY_L = \partial XY_R$ in p'. We use the notation

$$s' = g(p')$$

for the state s' generated by p'.

4.4. The readjustment process

The readjustment process can be looked upon as an algorithm which is used to compute a new state by a finite sequence of prestates $p_0, p_1, \ldots, p_N$. The sequence begins with a start p_0 and ends with a saturated prestate p_N.

The steps from one prestate to the next are arbitrary within some limits. Therefore the sequence is not uniquely determined. Nevertheless the last prestate of the sequence is always the same. Of course, this will have to be proven after the description of the readjustment process will be complete.

The steps of the readjustment process from prestate p_k to prestate p_{k+1} are the result of applying one of the three operations, or sometimes two of them one after the other to one directional. The steps belong to one of the following five categories, called **activities:**

1. Adaptation and confirmation of mature loose directionals
2. Dampening of univalued maladjusted non-zero tendencies
3. Adaptation of maladjusted tendencies
4. Confirmation of loose adjusted non-zero tendencies
5. Confirmation of loose adjusted zero tendencies

The first activity combines two operations, first adaptation and then confirmation of the same directional. Each of the other four activities involves only one operation. The activities are listed in their order of priority. Adaptation and confirmation of mature loose directionals has the highest priority, dampening of maladjusted univalued non-zero tendencies has the second highest priority, and so on.

Each activity is applied to a certain type of directional. We refer to this type as the **required type** for this activity. The readjustment process begins with the activity which has the highest priority among the activities for which at least one directional of the required type is available. An activity is continued as long as at least one directional of the type required for it is available. It does not matter which one of these directionals is chosen for the next step. An activity stops as soon as a prestate is reached at which no directionals of the required type for it are available. If the prestate is saturated, the readjustment process stops there, but otherwise it is continued with a new activity (the term “saturated” has been explained at the end of 4.4). The new activity is the activity with the highest priority among those for which at least one directional of the required type is available.

The definition of the readjustment process is now complete. The examination of the properties of this process will begin in 4.6. The remainder of this section will be devoted to the interpretation of the process.

For the sake of shortness we shall often refer to the k-th activity in the list as **activity** k. It may seem to be peculiar that activity 1 is the only one applied to current tendencies as well as system specific restrictions. The other four activities exclusively concern current tendencies. Later it will become clear that due to the anchoring requirement of 2.9 system specific restrictions are adapted and confirmed during an initial phase of the process in which activity 1 is applied. System specific restrictions are confirmed, before any other activity is taken up. Therefore these other activities only involve current tendencies.

The following two questions suggest themselves:

(a) Why does the process stick to one activity as long as possible?
(b) Why are the priorities of the activities chosen in this way?

We first look at question (a). The process should treat all directionals of the same type equally. We refer to this property as **neutrality**. Thus it should not matter which maladjusted tendency is adjusted first. The sequence, in which an activity is applied to the type of directionals required by it should not matter. One could achieve neutrality by simultaneously applying an operation or a combination of operations to all directionals of the same type. However, the theory proposed here aims at a reconstruction of boundedly rational qualitative reasoning on economic dynamics. From this point of view it is much more natural to apply an operation or a combination of operations to one directional at a time. We refer to this as the property of **step simplicity**. Sticking to the same activity as long as possible is important for achieving neutrality as well as step simplicity.

If one thinks of the readjustment process as an idealized picture of a quick dynamics which determines the transition from one state to the next, then neutrality seems to be an indispensable requirement. From this point of view one must look at all changes determined during one application of an activity as essentially simultaneous.

Admittedly there is a tension between the two interpretations of the readjustment process. On the one hand it is supposed to be a reasonable description of a quick dynamics and on the other hand it is a boundedly rational reasoning procedure. However, a reasonable theory should try to do justice to both interpretations. Therefore a prestate has been defined in such a way that it shows left and right values for every current tendency and a confirmation status for every directional. This together with the principle of sticking to one activity as long as possible enables the readjustment process to combine the substantial requirement of neutrality with the procedural one of step simplicity.

We now turn our attention to question (b). If a directional is mature the right hand side of the relevant confluence or restriction equation cannot be changed any more by the readjustment process. The final value of a directional is fully determined once it has become mature. Therefore it makes sense from the procedural point of view to give the first priority to activity 1.

In the course of the readjustment process a directional becomes mature as soon as the repercussions of the initial imbalance have reached a point at which the right hand side of the relevant confluence or restriction equation is fully determined. In terms of the interpretation as a quick dynamics, it is reasonable to suppose that such directionals move to their final value faster than others for whom the influences on the right hand side have not yet settled down.

In a situation in which there are no mature directionals it is important to remove the influence of all maladjusted non-zero tendencies before anything else is done. Therefore dampening of univalued maladjusted non-zero tendencies has the second highest priority. Of course, adaptation must come before confirmation. Therefore adaptation of maladjusted tendencies has the third highest priority.

Confirmation of adjusted non-zero tendencies cannot disturb the adjustment of other non-zero tendencies, but it may upset the adjustment of zero tendencies. This will be shown in Section 4.6. One may say that being adjusted is a robust property for non-zero tendencies but a fragile one for zero tendencies. Therefore confirmation of adjusted non-zero tendencies is given priority over confirmation of adjusted zero tendencies. This is reasonable in terms of both interpretations of the readjustment process.

4.5. The flow chart algorithm

The definition of the readjustment process in the preceding section needs to be complemented by a proof of the assertion that system specific restrictions are adapted and confirmed in a first phase of the process in which activity 1 is pursued. It also still needs to be shown that the process stops at a saturated prestate after a finite number of steps. Once these facts will have been established it still remains to be proven that the final saturated prestate is uniquely determined, even if this does not hold for the sequence of prestates leading to it.

The task of providing these proofs will be facilitated by an alternative description of the readjustment process. This alternative description is the flow chart of Figure 8. However, it is not obvious that the readjustment process fully agrees with the algorithm shown by Figure 8. Therefore we shall refer to it as the **flow chart algorithm.** It can then be proven that it is in fact nothing else than an alternative description of the readjustment process. We now proceed to explain more fully in what sense the readjustment process and the flow chart algorithm are equivalent.

A **realisation of the readjustment process** is a sequence $p_0, p_1, \ldots$ which begins with a start p_0 and conforms to the definition of the readjustment process of Section 4.4 as far as the sequence can be continued. The realisation may stop because it becomes impossible to continue the process or it may not stop at all. Eventually it will be proven that a realisation always ends with a saturated prestate, but this is not yet assumed by the definition of a realisation.

A **realisation of the flow chart algorithm** is defined in the same way as a sequence $p_0, p_1, \ldots$ beginning with a start p_0 and continued as long as possible according to the rules given by Figure 8. It is one of our goals to prove that a sequence $p_0, p_1, \ldots$ is a realisation of the readjustment process, if and only if it is a realisation of the flow chart algorithm. This is meant by saying that the flow chart algorithm is **equivalent** to the readjustment process.

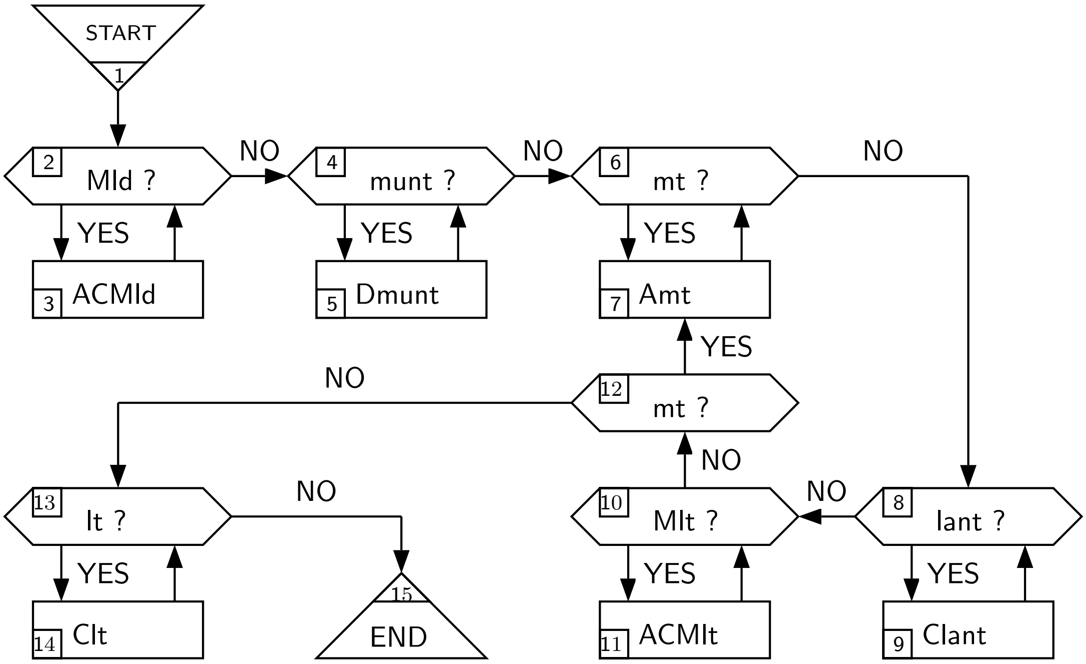
In Figure 8 start and end are indicated by triangles. A rhomboid represents a **switch** at which the question inside the rhomboid is asked. A rectangle represents an **operation**. For the sake of simplicity we also refer to an adaptation immediately followed by a confirmation as only one operation, even if it is a combination of two operations applied to the same directional.

Triangles, rhomboids and rectangles are numbered from 1 to 15. The numbers are shown inside the polygons. Switches and operations will be referred to by these numbers. In Figure 8 the activities correspond to pairs of a switch and an operation. At the switch the question is asked whether there are directionals of the type required for the activity. If the answer is YES then the activity is applied to one of these directionals. It is arbitrary which one of them is chosen if there are several such directionals. If the answer is NO then the algorithm moves to a new activity.

Activity 1 is represented by switch 2 and operation 3 and later again by 10 and 11. Activity 2 is pursued at 4 and 5, activity 3 at 6 and 7, and activity 4 at 8 and 9. It will be shown later that at switch 12 only adjusted zero tendencies can be loose. Therefore 13 and 14 represent activity 5.

In the same way as the readjustment process, the flow chart algorithm sticks to the same activity as long as possible. However, it remains to be shown that after the end of one activity the next one is the same in both procedures.

Notations: It will be convenient to introduce a notation for the prestates reached at NO-exits of switches. For a fixed realisation $p_0, p_1, \ldots$ of the flow chart algorithm let $r(k, m)$ be the prestate at which the question of switch k is answered by NO for the m-th time. The NO exits of switches $2, 4, 12$, and 13 can be reached only once but those of switches $6, 8$, and 10 can be passed many times. Note



Abbreviations:

a	adjusted	D	dampening	n	non-zero
A	adaptation	l	loose	u	univalued
C	confirmation	M	mature	t	tendency
d	directional	m	maladjusted		

The words "and" between A and C, "of a" after A, AC, C and D and the plural remain unexpressed.

Questions always begin with the unexpressed phrase "Are there any".

Examples:

Mld? stands for "Are there any mature loose directionals?"
ACMld stands for "Adaptation and confirmation of mature loose directionals".

FIGURE 8. Flowchart of the readjustment process

that the prestate at which the question of switch 12 is asked for the m-th time is the prestate $r(10, m)$. Only if at this prestate there are maladjusted tendencies $r(6, m+1), r(8, m+1)$ and $r(10, m+1)$ are reached. Otherwise $r(12, 1)$ is reached and the algorithm moves to 13 and 14 and finally to the end at 15. We call the $r(k, m)$ the **critical prestates** of the realisation $p_0, p_1, \ldots$.

In the remainder of this section two lemmas will be proven. The first one shows that at $r(2, 1)$ all system specific restrictions are firm. Therefore later applications of activity 1 can be restricted to the adaptation and confirmation of loose tendencies. The second lemma shows that the algorithm stops after a finite number of steps at triangle 15.

LEMMA 1. *All system specific restrictions are firm at the critical prestate* $r(2,1)$ *of every realisation* $p_0, p_1, \ldots$ *of the flow chart algorithm.*

PROOF. According to the anchoring requirement of 2.9 all system specific restrictions are anchored. Let S_0 be the set of all system pieces mentioned in part A of the recursive definition of "anchored" (see 2.9). For $k = 1, 2, \ldots$ let S_k be the set of all directionals such that the right hand side of the relevant confluence or restriction equation satisfies the condition that only system pieces in the sets $S_0, \ldots, S_{k-1}$ appear there, but with at least one of them in S_{k-1}. Obviously each anchored directional belongs to at least one of the S_k.

Suppose that a system specific restriction is in S_k. Then none of the $S_1, \ldots, S_{k-1}$ can be empty. Obviously the directionals in S_1 are mature at p_0. It can be seen immediately that activity 1 cannot stop before all anchored directionals have been adapted and confirmed. □

LEMMA 2. *Every realisation* $p_0, p_1, \ldots$ *of the flow chart algorithm ends with a last prestate* p_N.

PROOF. It is clear that at every switch the question asked there can be answered. If the answer is YES then the operation required by the algorithm can be performed. The algorithm is feasible in this sense and therefore cannot stop anywhere else than at triangle 15. However, we still have to exclude the possibility that the sequence $p_0, p_1, \ldots$ is infinite.

We first show that none of the activities can go on forever. Activities 1, 4, and 5 involve the confirmation of loose directionals. Each confirmation reduces the number of loose directionals and since there are only finitely many directionals in the system any activity involving confirmation has to stop after a finite number of steps.

Dampening is applied to maladjusted univalued non-zero tendencies. After dampening such a tendency is not univalued any more. Therefore the number of tendencies which can be dampened decreases with every dampening step. Therefore activity 2 stops after a finite number of steps.

Activity 3 applies the adaptation operation to maladjusted tendencies only. The application of this operation to one tendency never changes another adjusted tendency to a maladjusted one. This is due to the fact that only the values of left tendencies are changed by adaptation operations. An adjusted tendency remains adjusted at the present value of its left tendency as long as nothing is changed on the right hand side of its confluence. Therefore each step of activity 3 diminishes the number of tendencies to which it can be applied. Consequently activity 3 cannot go on forever.

Later it will be shown that activity 5 is pursued at 13 and 14. However, this fact is not yet available and we do not have to make use of it now. The activity at 13 and 14 involves confirmation and what has been said about activities $1, 4,$ and 5 applies here too.

In all cases the number of steps for which an activity can be continued is not only finite but bounded by the number of tendencies in the system. However this alone does not yet exclude the possibility that $p_0, p_1, \ldots$ is an infinite sequence. It could happen that switch 12 is reached infinitely often. In view of the structure of the flow chart this is the only possibility which is still open.

Suppose that switch 12 is reached and the answer to the question asked there is YES. Then the algorithm continues with activity 3 until all tendencies are adjusted. If then the answer to at least one of the questions in switch 8 and switch 10 is YES, then the number of firm tendencies is increased. Since the number of tendencies is finite it follows that this can happen only finitely often. Suppose that no tendencies are confirmed after switch 8 or switch 10 has been reached. If this happens the prestate $r(10, m)$ reached at the NO-exit of switch 10 is nothing else than the prestate $r(6, m)$. Therefore all tendencies are adjusted and all non-zero tendencies are firm at $r(10, m)$. Consequently the question of switch 12 is answered by NO and all loose tendencies are adjusted zero tendencies at the NO-exit of switch 12. These tendencies are confirmed by operation 14 and the end at triangle 15 is reached after a finite number of steps. Consequently the assertion of the lemma holds. □

4.6. Further properties of the flow chart algorithm

Section 4.5 has shown that the flow chart algorithm stops after a finite number of steps. Every realisation has the form $p_0, \ldots, p_N$. However it has not yet been shown that the state p_N is saturated in the sense that all directionals are adjusted and firm. This is important since otherwise the flow chart algorithm would fail to determine a new state. We shall now prove several lemmas which will lead to theorem 1. It is a part of the assertion of this theorem that p_N is saturated.

Theorem 1 implies that a new state is determined by any realization $p_0, \ldots, p_N$ of the flow chart algorithm beginning with an arbitrary start p_0. This has the consequence that the set of states cannot be empty. The example of a system violating the anchoring requirement discussed in 2.9 reveals that this is by no means trivial.

If beginning with an arbitrary start p_0 the flow chart algorithm is applied to the example of 2.9 then the realization becomes an infinite periodic cycle.

The anchoring requirement permits the proof of lemma 1. The system specific restrictions become firm at rectangle 3 in the first phase of activity 1. Only on this basis convergence of the flow chart algorithm has been obtained in 4.5.

LEMMA 3. *Let p_k be a prestate in a realisation $p_0, \dots, p_N$ of the flow chart algorithm and let ∂XY be a split tendency at p_k. Then $\partial XY_R = 0$ holds at p_k.*

PROOF. A maladjusted non-zero tendency can become a split tendency by dampening at 5. A maladjusted univalued zero tendency can become a split tendency by adaptation at 7. The flow chart shows that split tendencies cannot arise in any other way since the operations $3, 9, 11$ and 14 involve confirmation. Tendencies to which they are applied become univalued. The right tendency ∂XY_R of a split tendency is zero, regardless of whether the split is due to the dampening of a maladjusted non-zero tendency or the adaptation of a zero tendency. □

LEMMA 4. *Let $p_0, \dots, p_N$ be a realisation of the flow chart algorithm and in this sequence let p_k and p_{k+1}, be two consecutive prestates at which all system specific restrictions are firm. Moreover let ∂XY and ∂UV be two adjusted non-zero tendencies at p_k and assume that p_{k+1} results from p_k by confirmation of ∂XY. Then ∂UV remains adjusted at p_{k+1}.*

PROOF. Suppose that ∂UV_L has the value d at p_k with $d = +$ or $d = -$. Since ∂UV is adjusted at p_k the value of the right hand side of the confluence for ∂UV contains d. If ∂XY is univalued, then ∂XY_R is not changed by the confirmation of ∂XY and the value of the right hand side of the confluence for ∂UV also remains unchanged. Obviously in this case the assertion of the lemma holds.

Now assume that ∂XY is a split tendency. In view of lemma 3 we have $\partial XY_R = 0$ for p_k. Let f be the value of ∂XY_L at p_k. The confluence for ∂UV has the form

$$\partial UV = T \text{ @ } R.$$

If ∂UV is not subject to a restriction this is valid with $R = \{-, 0, +\}$. Let T_k and T_{k+1} be the values of T at p_k and p_{k+1} respectively. Since all system specific restrictions are firm, R has the same value at p_k and p_{k+1}. It follows by lemma 3 that we have $\partial XY_R = 0$. Moreover $\partial UV_L = d$ with $d = +$ or $d = -$ holds since ∂UV is an adjusted non-zero tendency at p_k. The value of ∂XY_R is changed from 0 to f with $f = +$ or $f = -$ by the confirmation of ∂XY. By assumption d is in T_k @ R. We have to distinguish two cases

(1) $T_k \cap R = \emptyset$
(2) $T_k \cap R \neq \emptyset$

Consider case 1). We must have $d \in R$ and R cannot have the value $\{-, 0, +\}$. Suppose we have $R = \{0, d\}$. In this subcase of 1) we would have $T_k = -d$ and therefore

$$T_k \mathbin{@} R = -d \mathbin{@} \{0, d\} = 0$$

contrary to the assumption that d is in $T_k \mathbin{@} R$. Therefore we must have $R = d$. Consequently d must be in $T_{k+1} \mathbin{@} R$ regardless of the value of T_{k+1}. Therefore the assertion holds in case 1).

Now consider case 2). In this case d belongs to T_k. Since T_k is a direction sum (see 2.4) it must have either the value d or the value $\{-, 0, +\}$. The change of the term ∂XY_R from zero to f may enlarge the value of T in the first case for $f = -d$ or it may leave it unchanged: In both cases d belongs to T_{k+1} and R. Therefore ∂UV is adjusted at p_{k+1} and the assertion of the lemma holds. □

Lemma 5. *Let $p_0, \dots, p_N$ be a realisation of the flow chart algorithm and in this sequence let p_k and p_{k+1} be two consecutive prestates, at which all system specific restrictions are firm. Let ∂XY be a loose mature zero tendency at p_k and assume that p_{k+1} results from p_k by adaptation and confirmation of ∂XY. Moreover let ∂UV be an adjusted non-zero tendency at p_k. Then ∂UV remains adjusted at p_{k+1}.*

Proof. In view of lemma 3 the tendency ∂XY is univalued at p_k. Suppose that at p_{k+1} we have $\partial XY_L = \partial XY_R = 0$, too. In this case nothing is changed on the right hand side of the confluence for ∂UV and the assertion holds. From now on we assume $\partial XY_L = \partial XY_R = f$ with $f = +$ or $f = -$ at p_{k+1}. As in the proof of lemma 4 let d with $d = +$ or $d = -$ be the value of ∂UV_L at p_k. The same cases 1) and 2) as in the proof of lemma 4 have to be distinguished. The remainder of the proof of lemma 5 is exactly the same as in the proof of lemma 4. □

Lemma 6. *Let $p_0, \dots, p_N$ be a realisation of the flow chart algorithm. At the critical prestate $r(8, 1)$ of $p_0, \dots, p_N$ all non-zero tendencies are adjusted. A non-zero tendency which is adjusted at $r(8, 1)$ remains adjusted at all later prestates of the realisation. Moreover at a critical prestate $r(10, m)$ of $p_0, \dots, p_N$ all non-zero tendencies are firm and adjusted.*

Proof. At $r(8, 1)$ all maladjusted tendencies, if there were any have been adapted by operation 7 and all loose adjusted non-zero tendencies have been confirmed by operation 9. There may be maladjusted tendencies at $r(8, 1)$ but these must be zero tendencies. Moreover all loose tendencies whether they are adjusted or not must be zero tendencies. This does not change as long as activity 1 is pursued at switch 10 and operation 11. Operation 11 may produce new adjusted and firm non-zero tendencies but in view of lemma 5 no adjusted non-zero tendency can become maladjusted by this. At $r(10, 1)$ all non-zero tendencies are adjusted and firm.

If then the question at switch 12 is answered by YES all maladjusted tendencies at $r(10,1)$ must be zero tendencies. Adaptation of these maladjusted zero tendencies by operation 7 will change them into new adjusted non-zero tendencies, but this will not disturb the adjustment of other non-zero tendencies, since only left tendencies are changed by the adaptation operation. In view of lemma 4 the confirmations at operation 9 also do not disturb the adjustment of other non-zero tendencies. These confirmations may lead to new maladjusted zero tendencies, but not to new maladjusted non-zero tendencies.

At $r(8,m)$ with $m > 1$ the situation is essentially the same as at $r(8,1)$. This can be seen by a simple induction argument. Therefore at $r(10,m)$ with $m > 1$ all non-zero tendencies are adjusted and firm as asserted by the lemma and no adjusted non-zero tendency has become maladjusted up to then.

When finally the question at switch 12 is answered by NO, all loose tendencies are adjusted zero tendencies. Their confirmation by operation 14 cannot lead to any new maladjusted tendencies and cannot disturb the adjustment of adjusted ones, since nothing is changed on the right hand side of confluences. This completes the proof of the lemma. □

REMARK. *It is a consequence of the lemma that activity 5 is pursued at switch 13 and operation 14.*

LEMMA 7. *Let $p_0, \ldots, p_N$ be a realisation of the flow chart algorithm. Every system specific restriction $\Box XY$ which is loose at a prestate p_k of this sequence and adjusted and firm at the next prestate p_{k+1} remains adjusted and firm at each of the prestates $p_{k+1}, \ldots, p_N$ and has the same value at all these prestates.*

PROOF. In view of lemma 1 every system specific restriction is firm at $r(2,1)$. Consider a system specific restriction $\Box XY$. At the last prestate p_k at which $\Box XY$ is loose, it is mature and $\Box XY$ is adapted and confirmed by operation 3 in the step from p_k to p_{k+1}. The right hand side of the restriction equation for $\Box XY$ and the value of $\Box XY$ cannot be changed by later operations, since it is fully determined by directionals which are firm at p_k and later operations concern only loose directionals. This shows that the assertion of the lemma is true. □

LEMMA 8. *Let $p_0, \ldots, p_N$ be a realisation of the flow chart algorithm. Every tendency ∂XY which is loose at a prestate p_k of this sequence and firm at the next prestate p_{k+1} is adjusted and firm at each of the prestates $p_{k+1}, \ldots, p_N$ and has the same value at all these prestates.*

PROOF. Consider first the case that ∂XY is adapted and confirmed in the step from p_k to p_{k+1}. In this case ∂XY is mature at p_k. This means that the right hand side of the confluence for ∂XY is fully determined by directionals which are firm at p_k and cannot be changed by later operations which concern only loose directionals. For the same reason the value of ∂XY remains unchanged by later operations. In the case considered up to now the assertion of the lemma holds.

Consider the case that a loose adjusted non-zero tendency ∂XY is confirmed by operation 9. It follows by lemma 4 that confirmations of other non-zero tendencies neither disturb the adjustment nor change the value of ∂XY. Lemma 5 shows that the same is true for later applications of operation 11 to mature loose zero tendencies. Lemma 6 shows that only zero tendencies can be maladjusted after switch 8 has been left by the NO-exit for the first time. Moreover all non-zero tendencies have been confirmed by operation 9 when this happens. Therefore operation 11 is only applied to zero tendencies. The confirmation of adjusted zero tendencies by operation 14 does not influence the right hand side of the confluence for ∂XY either. We can conclude that the assertion also holds for adjusted non-zero tendencies confirmed by operation 9.

It remains to look at the case of loose tendencies confirmed by operation 14. After switch 8 is left by the NO exit all non-zero tendencies are confirmed and no new loose ones can be produced by operation 11. Therefore at the NO-exit of switch 12 all loose tendencies are adjusted zero tendencies. The values of right hand sides of confluences are not changed by the confirmations of such tendencies. Therefore the assertion of the lemma holds in this case, too. □

THEOREM 1. *Let $p_0, \ldots, p_N$ be a realisation of the flow chart algorithm. Then p_N is saturated (see Section 4.4). Moreover if a directional is for the first time firm at a prestate p_k then it is adjusted and firm at all prestates $p_k, \ldots, p_N$ and has the same value at all these prestates.*

COROLLARY 1. *Every base B has at least one state. Moreover for every specification of values of scaled variables and lagged tendencies a base always has at least one state with the specified values for these components.*

PROOF. In order to prove the assertion of the theorem (not the corrolary) it is sufficient to show that every directional becomes firm at some prestate of the sequence $p_0, \ldots, p_N$. The remainder follows by lemma 7 and lemma 8.

It is a consequence of lemma 1 that each system specific restriction becomes firm at some prestate. According to lemma 8 all non-zero tendencies are firm at a critical prestate of the form $r(10, m)$. Consider the prestate $r(12, 1)$. Since all non-zero tendencies are adjusted and firm at $r(12, 1)$, any loose tendencies at this prestate must be adjusted zero tendencies. All these tendencies are then confirmed by operation 14. It remains to prove the corrolary.

For a given specification of scaled variables and lagged tendencies let p_0 be the prestate with the specified values for these components for which all left and right tendencies and all system specific restrictions have the value zero. Obviously p_0 is a start. Let $p_0, \ldots, p_N$ be the realization of the flow chart algorithm beginning with p_0. It has been shown that p_N is saturated. Therefore B has the state generated by p_N (see 4.3). This completes the proof of the corrolary. □

COMMENT. *In 2.9 an example of a structure was presented which has all the properties of a base with the exception of the anchoring requirement. As we have seen this structure has no states. The corrolary of theorem 1 shows that the inclusion of the anchoring requirement into the definition of a base guarantees the existence of states.*

4.7. Equivalence of the readjustment process and the flow chart algorithm

The results of the preceding section show that the flow chart algorithm is feasible in the sense that it stops after a finite number of steps at a saturated prestate. However, it is not yet clear whether the flow chart algorithm is equivalent to the readjustment process in the sense that the two procedures have the same realizations. Theorem 2 will give a positive answer to this question.

THEOREM 2. *The set of all realisations of the readjustment process is the set of all realisations of the flow chart algorithm.*

PROOF. As has been pointed out before the flow chart algorithm as well as the readjustment process stick to an activity as long as possible. It is sufficient to show that after the end of an activity the two procedures begin with the same new activity. In order to do this we will look at all points of the flow chart at which a new activity may begin. It will be argued that a new activity is always the one chosen by the readjustment process at the same prestate.

Consider a realisation $p_0, \ldots, p_N$ of the flow chart algorithm and the critical prestates $r(k, m)$ of this realisation. If there are mature directionals at p_0 then activity 1 is chosen at p_0 by both procedures. Otherwise we have $p_0 = r(2, 1)$. At $r(2, 1)$ there are no mature directionals and activity 2 is chosen by both procedures, if there are maladjusted non-zero tendencies there. Otherwise we have $r(2, 1) = r(4, 1)$. Since new mature directionals can only arise by confirmations of other directionals, there are no mature directionals at $r(4, 1)$. Moreover there are no univalued maladjusted non-zero tendencies. Activities 1 and 2 are not applicable there and activity 3 is chosen by both procedures, if there are maladjusted tendencies at this prestate. Otherwise we have $r(4, 1) = r(6, 1)$. At $r(6, 1)$ activities 1, 2, and 3 are not applicable and both procedures move to activity 4 if there are loose adjusted non-zero tendencies at $r(6, 1)$. Otherwise we have $r(6, 1) = r(8, 1)$.

As long as there are no new confirmations there cannot be any new mature loose tendencies. Therefore the opportunity for pursuing activity 1 does not arise at $r(4,1)$ and $r(6,1)$. However the confirmations at operation 9 may have produced new mature loose tendencies at $r(8,1)$. If this is the case then both procedures choose activity 1 at $r(8,1)$. Otherwise we have $r(8,1) = r(10,1)$.

Consider a critical prestate of the form $r(10,m)$ reached by the realisation $p_0,\ldots,p_N$. It is clear that activity 1 is not applicable there. It follows by lemma 6 that activity 2 is not applicable at $r(10,m)$, since all non-zero tendencies are adjusted and firm at this prestate. There may be maladjusted tendencies at $r(10,m)$ but these must be zero-tendencies. If this is the case then activity 3 is chosen by both procedures at $r(10,m)$. Otherwise we have $r(10,m) = r(12,1)$ and the NO-exit of switch 12 is reached.

Suppose that activity 3 follows $r(10,m)$. Then the next critical prestate is $r(6,m+1)$. As at $r(6,1)$ activities 1, 2, and 3 are not applicable at $r(6,m+1)$ and activity 4 is chosen there by both procedures, if there are loose adjusted non-zero tendencies at $r(6,m+1)$. Otherwise $r(6,m+1) = r(8,m+1)$ holds. At $r(8,m+1)$ both procedures choose activity 1, if new mature loose tendencies have been produced by the confirmations during activity 4. Otherwise we have $r(8,m+1) = r(10,m+1)$.

It follows by induction on m that both procedures move to the same new activity at all $r(k,m)$ with $k = 6, 8$, and 10 as long as $r(12,1)$ is not reached. At $r(12,1)$ all loose tendencies are adjusted zero tendencies and both procedures move to activity 5. This completes the proof. □

4.8. Order independence

In the preceding section it has been shown that there is no difference between the flow chart algorithm and the readjustment process. Therefore we can drop the distinction between the two procedures. In the remainder of this book we shall not talk about the flow chart algorithm any more, but only about the readjustment process. However, the flow chart of Figure 8 is a more convenient description of this process than the definition of 4.4. The flow chart embodies some properties which are not apparent from the definition, e.g. the important result that the activity of dampening maladjusted univalued tendencies can be pursued only once in a realisation of the process. We shall continue to make use of the notion of the critical prestates $r(k,m)$ of a realisation $p_0,\ldots,p_N$. In view of theorem 2 we may think of $p_0,\ldots,p_N$ as a realisation of the readjustment process.

It still needs to be proven that the order in which an activity is applied to directionals does not matter as long as it is compatible with the definition of the readjustment process. This is what is meant by the term order independence. It will be shown that every realisation of the readjustment process starting with the same start has the same critical prestates and leads to the same final saturated prestate.

LEMMA 9. *Every realisation* $p_0, \ldots, p_N$ *of the readjustment process with the same start* p_0 *has the same critical prestate* $r(2,1)$.

PROOF. If there are no mature directionals at p_0 we have $p_0 = r(2,1)$. Obviously the assertion holds in this case. From now on assume that at least one directional is mature at p_0. We now recursively define a sequence $q_0, q_1, \ldots$ of prestates and a sequence $D_1, D_2, \ldots$ of sets of directionals. The prestate q_0 is the start p_0. For $k = 1, 2, \ldots$ the set D_k is the set of all loose mature directionals at q_{k-1} and q_k is the prestate which results by adaptation and confirmation of all directionals in D_k from q_{k-1}. It is clear that the order of these adaptations and confirmations does not matter.

For some positive integer K we must have $D_K \neq \emptyset$ and $D_{K+1} = \emptyset$ since there are only finitely many directionals which can be confirmed. Let D be the union of the $D_1, \ldots, D_K$. Obviously the sets $D_1, \ldots, D_K$ form a partition of D. At q_K all directionals in D are firm and all other directionals are loose. Obviously q_K is the critical prestate $r(2,1)$ for some realizations of the readjustment process, namely those in which activity 1 is applied first to the directionals in D_1, then to those in D_2 and so on. Let $p'_0, \ldots, p'_N$ with $p'_0 = p_0$ be a realization of this kind.

We show by induction on k that each directional ∂XY or $\Box XY$ in D_k is adapted and confirmed at the same value

$$t_{XY} = \partial XY_L = \partial XY_R$$

or

$$R_{XY} = \Box XY$$

in every realization $p_0, \ldots, p_N$ of the readjustment process. The assertion holds for $k = 1$, since for a directional in D_1 the right hand side of the confluence or the restriction equation is fully determined at q_0 and cannot change any more by later adaptations and confirmations of other loose mature directionals.

Assume that the assertion holds for $k = 1, \ldots, s$. We shall show that then it also holds for $k = s + 1$. Consider a directional ∂XY or $\Box XY$ in D_{s+1}. Let $p_0, \ldots, p_M$ be a realization of the readjustment process, different from $p'_0, \ldots, p'_N$. Moreover let $r'(2,1)$ be the prestate which is reached in $p'_0, \ldots, p'_N$ at the NO - exit of switch 2 and let L be the set of all directionals on the right hand side of the confluence for ∂XY or the restriction equation for $\Box XY$ which are firm at q_s. A directional which is firm at q_s is in one of the sets $D_1, \ldots, D_s$. The assertion holds for such directionals. Therefore the directionals in L have the same values at $r(2,1)$ and $r'(2,1)$. These values fully determine the value of the right hand side of the confluence for ∂XY or the restriction equation for $\Box XY$. This right hand side had already the same value when ∂XY or $\Box XY$ was adapted and confirmed in $p_0, \ldots, p_M$ or in $p'_0, \ldots, p'_N$. Therefore the values t_{XY} and t'_{XY} of ∂XY or the values R_{XY} and R'_{XY} of $\Box XY$ in $r(2,1)$ and $r'(2,1)$, respectively, must be equal. Consequently the assertion also holds for $k = s + 1$.

We can conclude that at the critical prestate $r(2,1)$ of every realization $p_0, \ldots, p_N$ of the readjustment process beginning with the same start p_0 a directional in D always has the same value. The directionals in D are firm and those outside D are loose at $r(2,1)$. Only tendencies can be outside D at $r(2,1)$. The application of activity 1 to directionals in D has no influence on the left and right values of tendencies outside D. Therefore the critical prestate $r(2,1)$ is the same one for every realization $p_0, \ldots, p_N$ of the readjustment process beginning with the same start p_0. □

LEMMA 10. *Let p_k be a prestate and let ∂XY be a maladjusted univalued non-zero tendency at p_k and let p_{k+1} be the prestate which results from p_k by dampening ∂XY. Moreover let ∂UV be a maladjusted non-zero tendency at p_k different from ∂XY. Then ∂UV remains maladjusted at p_{k+1}.*

PROOF. Note that ∂UV may be univalued or split at p_k. Let

$$\partial UV = T \ @ \ R$$

be the confluence for ∂UV. In the case that ∂UV is not subject to any restriction the confluence has this form for $R = \{-, 0, +\}$. Let T_k and T_{k+1} be the values of T at p_k and p_{k+1}, respectively. ∂XY_R has the value d with $d \neq 0$ at p_k and the value 0 at p_{k+1}. If ∂XY does not appear in T then we have $T_k = T_{k+1}$. Assume that ∂XY appears in T. Since T is a direction sum, T_k must have one of the values $-, 0, +$, and $\{-, 0, +\}$. In the case $T_k = +$ or $T_k = -$ we may have $T_{k+1} = 0$. For $T_k = \{-, 0, +\}$ the change of ∂XY_R from d to 0 may result in $T_{k+1} = +$ or $T_{k+1} = -$. Table 18 shows the possibilities for T_k and T_{k+1}.

		T_{k+1}			
		$-$	0	$+$	$\{-,0,+\}$
T_k	$-$	YES	YES	NO	NO
	0	NO	YES	NO	NO
	$+$	NO	YES	YES	NO
	$\{-,0,+\}$	YES	NO	YES	YES

TABLE 18. Possibilities for T_k and T_{k+1} in the proof of Lemma 10

Let g be the value of ∂UV_L at p_k. Dampening ∂XY does not change ∂UV_L. Therefore g is also the value of ∂UV_L at p_{k+1}. Since ∂UV is a non-zero tendency at p_k we have $g \neq 0$.

A case distinction will be made according to the value of R. First consider the case $R = \{-, 0, +\}$. In this case g cannot belong to T_k, since ∂UV is maladjusted at p_k. We must have $T_k = -g$ or $T_k = 0$. It follows by Table 18, that for $T_k = -g$ we can have $T_{k+1} = -g$ or $T_{k+1} = 0$ and for $T_k = 0$ only $T_{k+1} = 0$. Therefore g is not an element of T_{k+1} and ∂UV is maladjusted at p_{k+1}.

Now consider the case that R has only one element, a direction f. Since ∂UV is maladjusted at p_k it follows that $g \neq f$ holds. Therefore ∂UV is maladjusted at p_{k+1}, too, regardless of the value of T_{k+1}.

We now look at the remaining two cases for R:

$$R = \{0, g\} \qquad \text{and} \qquad R = \{-g, 0\}.$$

Consider the case $R = \{0, g\}$. In this case T_k cannot contain g. We must have $T_k = -g$ or $T_k = 0$. In both cases T_k @ R has the value zero. It follows by Table 18 that for $T_k = 0$ we have $T_{k+1} = 0$ and for $T_k = -g$ either $T_{k+1} = -g$ or $T_{k+1} = 0$. This has the consequence that

$$T_{k+1} \text{ @ } R = 0$$

is true for all possible values of T_{k+1}. Obviously ∂UV is maladjusted at p_{k+1} in this case.

Now consider the case $R = \{-g, 0\}$. Obviously g cannot be in T_{k+1} @ R regardless of the value of T_{k+1}. Therefore ∂UV is maladjusted at p_{k+1} in this case, too. We can conclude that the assertion of the lemma holds. □

LEMMA 11. *Every realization $p_0, \ldots, p_N$ of the readjustment process beginning with the same start p_0 has the same critical prestate $r(4, 1)$.*

PROOF. In view of lemma 9 we can restrict our attention to the dampening steps between $r(2, 1)$ and $r(4, 1)$. Lemma 10 has shown that a maladjusted non-zero tendency remains maladjusted if another one is dampened. However, it can happen that an adjusted non-zero tendency becomes maladjusted if another one is dampened.

Let D_1 be the set of all maladjusted univalued non-zero tendencies at $r(2, 1)$ and let z_1, be the prestate which results from $r(2, 1)$ by dampening all tendencies in D_1, one after the other. It follows by lemma 10 that the order in which activity 2 is applied to the tendencies in D_1 does not matter. z_1 does not depend on this order.

For $k = 2, 3, \ldots$ let D_k be the set of all univalued maladjusted non-zero tendencies which are maladjusted at z_{k-1}. For some positive integer K, the set D_K

will be empty, since there are only finitely many tendencies. Let D be the union of all D_k with $k = 1, \ldots, K-1$. It is clear that at $r(4,1)$ all tendencies in D and no others have been dampened. By lemma 10 the order in which this happened does not matter. It follows that the critical prestate $r(4,1)$ is always the same. This completes the proof of the lemma. □

LEMMA 12. *Let p_0 be a start. Every realization $p_0, \ldots, p_N$ of the readjustment process has the same critical prestate $r(6,1)$.*

PROOF. If there are no maladjusted tendencies at $r(4,1)$ then we have $r(6,1) = r(4,1)$ and the assertion of the lemma holds. Therefore from now on we assume that there are maladjusted tendencies at $r(4,1)$. In view of lemma 9 and lemma 11 we can restrict our attention to the adaptation steps between $r(4,1)$ and $r(6,1)$.

Adaptation changes left tendencies only and therefore has no influence on the right hand side of confluences of other variables. On the way from $r(4,1)$ to $r(6,1)$ all tendencies which are maladjusted at $r(4,1)$ become adjusted. The tendencies which are adjusted at $r(4,1)$ remain adjusted. The order in which activity 3 is applied to the tendencies which are maladjusted at $r(4,1)$ does not matter. The assertion of the lemma is true. □

LEMMA 13. *Every realization $p_0, \ldots, p_N$ of the readjustment process with the same critical prestate $r(6,m)$ reaches the same critical prestate $r(8,m)$. This is true for all $m = 1, 2, \ldots$ such that $p_0, \ldots, p_N$ has a critical prestate $r(6,m)$.*

PROOF. At $r(6,m)$ all tendencies are adjusted. It follows by lemma 4 that a confirmation of a loose adjusted non-zero tendency by operation 9 does not disturb the adjustment of other adjusted non-zero tendencies. Loose adjusted non-zero tendencies at $r(6,m)$ are split. This can be seen as follows. Some of these tendencies become adjusted univalued zero tendencies by operation 7 and for others the value of the left tendency may change to the opposite one. These tendencies remain split after adaptation. Maladjusted zero tendencies become adjusted split non-zero tendencies when operation 7 is applied to them.

On the way from $r(6,m)$ to $r(8,m)$ no adjusted non-zero tendency can become maladjusted. However, if a loose split adjusted non-zero tendency is confirmed, its right tendency changes from zero to a non-zero value. Thereby an adjusted loose zero tendency may become maladjusted. All loose adjusted non-zero tendencies become firm on the way to $r(8,m)$ and new ones cannot arise. If thereby a loose adjusted zero tendency becomes maladjusted, this happens to this tendency for any order in which activity 4 is applied to the loose adjusted non-zero tendencies at $r(6,m)$. We can conclude that $r(8,m)$ does not depend on this order. The proof of lemma 13 is now complete. □

LEMMA 14. *Every realization $p_0, \ldots, p_N$ of the readjustment process with the same critical prestate $r(8, m)$ reaches the same critical prestate $r(10, m)$. This is true for all $m = 1, 2, \ldots$ such that $p_0, \ldots, p_N$ has a critical prestate $r(8, m)$.*

PROOF. Activity 1 is pursued on the way from $r(8, m)$ to $r(10, m)$. The proof of lemma 9 can be transferred to this situation without any difficulty. Instead of looking at the section $p_0, \ldots, r(2, 1)$ of a realization of the readjustment process we now have to look at a later section $r(8, m), \ldots, r(10, m)$. The proof becomes simpler since all system specific restrictions are firm at $r(8, m)$ and only mature loose tendencies need to be considered. However it is not necessary to work this out in detail. □

LEMMA 15. *Every realization $p_0, \ldots, p_N$ of the readjustment process with the same critical prestate $r(10, m)$ reaches the same critical prestate $r(6, m+1)$ if there are maladjusted tendencies at $r(10, m)$.*

PROOF. Assume that there is at least one maladjusted tendency at $r(10, m)$. Then at $r(10, m)$ the readjustment process moves to the new activity 3. The situation is essentially the same as in the section between $r(4, 1)$ and $r(6, 1)$. What has been said about the effects of adaptation steps in the proof of lemma 12 applies also here. We conclude that the order does not matter, in which the maladjusted tendencies at $r(10, m)$ are adapted. At the end of activity 3 always the same critical prestate $r(6, m + 1)$ is reached. The assertion of the lemma is true. □

THEOREM 3. *Every realization $p_0, \ldots, p_N$ of the readjustment process beginning with the same start p_0 has the same critical prestates and the same final prestate p_N.*

PROOF. It follows by the lemmas 9, 11 and 12 that the assertion holds for $r(2, 1)$, $r(4, 1)$ and $r(6, 1)$. Lemma 13 and lemma 14 show that the same is true for $r(8, 1)$ and $r(10, 1)$. Moreover a simple induction argument based on the lemmas 15, 13, and 14 extends the result to all critical prestates of the form $r(6, m)$, $r(8, m)$ and $r(10, m)$ where $m = 2, 3, \ldots$ is an integer such that there are maladjusted tendencies at $r(10, m - 1)$.

As soon as there are no maladjusted tendencies at $r(10, m)$ we have $r(10, m) = r(12, 1)$. It follows by lemma 6 that only adjusted zero tendencies can be loose at $r(12, 1)$. The confirmations at operation 14 do not change the values of left and right tendencies. Nothing else than the confirmation status of adjusted zero tendencies is changed on the way from $r(12, 1)$ to the end at triangle 15. It is clear that the order does not matter, in which activity 5 is applied to the loose adjusted tendencies. Therefore the assertion of the theorem is true. □

REMARK. *We did not show that the number $N+1$ of prestates is the same one in every realization $p_0, \ldots, p_N$. However, this is actually the case. Since $r(2,1)$ is always the same prestate, the same number of adaptations and confirmations must take place in every realization between p_0 and $r(2,1)$. This argument applies to any two consecutive critical prestates. As long as one activity is performed every application of this activity is irreversible and the difference between the two prestates determines the number of operations.*

4.9. Main transitions

The procedures used for the determination of the result of a main transition have been discussed in 3.2 and 4.3, but these explanations preceded the definition of the readjustment process. Therefore, we have to recapitulate what has been said before and to fill in details which may still be unclear. In this section attention is restricted to main transitions. We shall look at perturbances in connection with the definition of stability in chapter 5. In the following a case distinction between reanchorings, i.e. shifts or lag extinctions, and other transition causes will be made.

Reanchorings: In the case of a shift $\omega = [XY \to v]$ or $\omega = [XY \to V]$ or a lag extinction $[\partial XY^-]$ at a state s the readjustment process is applied in the original system Φ. Beginning with the transition start $p_0 = p_0(\omega, s)$ a realization $p_0, \ldots, p_N$ of the readjustment process is constructed following the flow chart of Figure 8. The final prestate p_N does not depend on the particular realization chosen (theorem 3) and is saturated. The new state reached by the transition is the state

$$s' = g(p_N)$$

For the definition of the function g see 4.3.

Tendency switches: Let $\omega = [\partial XY \to d_2]$ be a tendency switch of ∂XY from d_1 to d_2 at a state s. The first step of the procedure (a second step may have to follow) is the construction of a realization of the readjustment process in the hypothetical base B_ω beginning with $p_0 = p_0(s)$. In the hypothetical base B_ω the confluence for ∂XY is replaced by

$$\partial XY = d_2$$

and nothing else in the base $B = (\Lambda, \Gamma)$ of Φ is changed (see 3.2.3). Consider a realization $p_0, \ldots, p_N$ in B_ω. The final prestate p_N does not depend on the particular realization chosen. Moreover p_N is saturated in B_ω. This means that p_N satisfies all confluences and restriction equations of B with the possible exception of the confluence for ∂XY. If p_N also satisfies the confluence for ∂XY in B, then p_N is saturated in B and the new state reached by the tendency switch of ∂XY

from d_1 to d_2 is given by

$$s' = g(p_N)$$

In the following it will be assumed that p_N does not satisfy the confluence for ∂XY in B. In this case the tendency switch ω of ∂XY from d_1 to d_2 is not feasible. If either $d_1 = 0$ or $d_2 = 0$ holds then ω is not only not feasible but infeasible. In this case no transition is caused by the switch ω. There is no edge of the tentative transition diagram associated to ω. With this conclusion the investigation of ω ends after the first step.

Now assume that ω is not feasible and $d_1 \neq 0$ and $d_2 \neq 0$ hold. Then ω is a switch from $-$ to $+$ or from $+$ to $-$. In order to find out whether ω is semifeasible, we have to look at the halfway switch μ of ∂XY from d_1 to zero at s. In the hypothetical base B_μ for this halfway switch the confluence for ∂XY is replaced by

$$\partial XY = 0$$

and nothing else in the base B of Φ is changed. A realization $p_0, \ldots, p_M$ of the readjustment process in B_μ beginning with $p_0 = p_0(s)$ is constructed. The final prestate p_M does not depend on the particular realization chosen and is saturated in B_μ. If p_M also satisfies the confluence for ∂XY in B, then ω is semifeasible and the new state reached by ω is given by

$$s' = g(p_M).$$

If p_M fails to satisfy the confluence for ∂XY in B then ω is infeasible and causes no transition. In this case the tentative transition diagram has no edge associated to ω.

It will be shown in chapter 5 that immediate tendency switches are always feasible. This facilitates the analysis of systems in which no tardy tendency switches have rank 1 at any state.

4.10. Examples of readjustment process realizations

4.10.1. The upswing of the model of Table 4. Table 19 shows realizations of the readjustment process for all transitions in the upswing of the cycle of the model of Table 4, from the lower turning point b to the upper turning point c (see Figure 3 in 2.5 and Table 13 in 3.8). The downswing is not presented here, since it is a "mirror image" of the upswing with $-$ and $+$ interchanged, not only in states, but also in the prestates of the readjustment process realizations.

Table 19 follows the following **conventions for readjustment process tables** which will also be used for other tables: The rows describe states or prestates. In the case of a state the first column with the heading "comments" indicates which state it is. The time order is from above to below. Horizontal lines separate states

Comment	PD	$\Box DE = \triangleright PD$	∂PD	∂DE	∂IN	activity
state 1	b	$\{0,+\}$	$+$	$+$	$-$	
	L	$\{0,+\}$	$++$	$++$	$--$	
		$\{-,0,+\}$F				1
$[PD \to L]$					$--$F	1
			$++$F			4
				$++$F		4
state 2	L	$\{-,0,+\}$	$+$	$+$	$-$	
	n	$\{-,0,+\}$	$++$	$++$	$--$	
		$\{-,0,+\}$F				1
$[PD \to n]$					00 F	1
			$++$F			4
				$++$F		4
state 3	n	$\{-,0,+\}$	$+$	$+$	0	
	H	$\{-,0,+\}$	$++$	$++$	00	
		$\{-,0,+\}$F				1
$[PD \to H]$					$++$ F	1
			$++$F			4
				$++$F		4
state 4	H	$\{-,0,+\}$	$+$	$+$	$+$	
	c	$\{-,0,+\}$	$++$	$++$	$++$	
		$\{-,0\}$F				1
$[PD \to c]$					$++$ F	1
			$+0$			2
				$+0$		2
			00			3
				-0		3
				$--$F		4
			$--$F			1
state 5	c	$\{-,0\}$	$-$	$-$	$+$	

TABLE 19. The readjustment process for the transitions in the upswing of the model of Table 4

from the readjustment processes before and after them. A readjustment process begins with the transition start associated to the state above it and the transition cause shown under comments beside it.

The table has a column for each component of a state. In a row describing a state an entry in one of these columns is the value of the associated component. In a row describing a prestate the same is true for values of scaled variables or lagged tendencies. An F at the right of a field in a column for a directional indicates that the directional is firm at the prestate described by the row. In such a row the two entries in a column for a current tendency refer to the values of the left tendency (the first entry) and to the value of the right tendency (the second entry). The values of system specific restrictions and left and right tendencies are not shown in the rows following the one for the transition start, unless something has changed in the column. Similarly the entry F is shown only once in a column. The last column with the heading "activity" shows which activity has been applied in the step of the readjustment process from the preceding prestate to the current one.

After these general explanations of the conventions for readjustment process tables we now turn our attention to the specific example of Table 19. In the model of Table 4 the system specific restriction and ∂IN are anchored. Therefore these directionals are adjusted and at the beginning of each of the realizations shown by Table 19. In the first three realizations between states 1 and state 4, the other two tendencies are adjusted non-zero tendencies after these first two steps and are then confirmed by activity 4.

The transition from state 4 to state 5 is slightly more involved. After the first two steps of the readjustment process, the tendencies ∂PD and ∂DE are maladjusted non-zero tendencies. They are dampened by activity 2. Thereby the value of the right hand side of the confluence of ∂DE becomes negative. After the adaptation of the two tendencies by activity 3 the tendency ∂DE becomes an adjusted non-zero tendency. Therefore ∂DE is confirmed by activity 4. Thereby ∂PD becomes mature and is adapted and confirmed by activity 1.

We know by Table 5 that there is only one state with $PD = c$. One does not need the readjustment process in order to determine the end result of the transition caused by the shift of PD from H to c at state 4. However, the readjustment process provides a dynamic picture of what happens at the upper turning point.

4.10.2. The upswing of the model of Table 6. Table 20 shows the upswing of the cycle for the model of Table 6 from the lower turning point at state 3 to the upper turning point at state 19 (see Figure 4 in 2.10 and Table 14 in 3.8.3).

Comment	PD	∂PD^-	$\square DE$	∂PD	∂DE	∂IN	activity
state 3	b	0	$\{0,+\}$	$+$	$+$	$-$	
							— continued next page

Table 20: The upswing of the model of Table 6

Comment	PD	∂PD^-	$\Box DE$	∂PD	∂DE	∂IN	activity
	L	0	$\{0,+\}$	++	++	$--$	
			$\{-,0,+\}$F				1
$[PD \to L]$						$--$F	1
					++F		1
				++F			1
state 8	L	0	$\{-,0,+\}$	+	+	$-$	
	L	+	$\{-,0,+\}$	++	++	$--$	
			$\{-,0,+\}$F				1
$[\partial PD^-]$						$--$ F	1
					++F		1
				++F			1
state 9	L	+	$\{-,0,+\}$	+	+	$-$	
	n	+	$\{-,0,+\}$	++	++	$--$	
			$\{-,0,+\}$F				1
$[PD \to n]$						00 F	1
					++F		1
				++F			1
state 12	n	+	$\{-,0,+\}$	+	+	0	
	H	+	$\{-,0,+\}$	++	++	00	
			$\{-,0,+\}$F				1
$[PD \to H]$						++ F	1
					++F		1
				++F			1
state 17	H	+	$\{-,0,+\}$	+	+	+	
	c	+	$\{-,0,+\}$	++	++	++	
			$\{-,0\}$F				1
$[PD \to c]$						++ F	1
					00F		1

— *continued next page*

Table 20: The upswing of the model of Table 6

Comment	PD	∂PD^-	$\Box DE$	∂PD	∂DE	∂IN	activity
				00F			1
state 21	c	$+$	$\{-,0\}$	0	0	$+$	
	c	0	$\{-,0\}$	00	00	$++$	
			$\{-,0\}$F				1
$[\partial PD^-]$						$++$ F	1
					$--$F		1
				$--$F			1
state 19	c	0	$\{-,0\}$	$-$	$-$	$+$	
							— *continuation*

Table 20: The upswing of the model of Table 6

The conventions for readjustment process tables explained in 4.10.1 are valid for this table. Only the upswing is shown since here, too, the downswing is the "mirror image" of the upswing.

In the model of Table 6 all directionals are anchored. Therefore only activity 1 is used in the readjustment process. The directionals can always be adapted and confirmed in the same order: $\Box DE, \partial IN, \partial DE, \partial PD$. It is clear that for each transition the transition cause is the one with the highest priority according to the general principles of 4.10.2 and Table 14.

4.10.3. The tendency switch of ∂DE at state 4 of the model of Table 4. In 3.8.2 an alternative priority ranking for the model of Table 4 has been described. This priority ranking gives rank 1 to the tendency switch $\omega = [\partial DE \to -]$ at state 4. A heuristic discussion of this switch has been presented in 3.2. Table 21 shows a realization of the readjustment process for the transition caused by this switch in the hypothetical base B_ω beginning with the transition start for ω at state 4. The conventions for readjustment process tables are valid for this table, but complemented by a first column which indicates whether a state or a readjustment process realization belongs to the hypothetical system base or to the original one.

After state 4 a realization of the readjustment process in the hypothetical base is shown by the table. This realization begins with the transition start for $\omega = [\partial DE \to -]$ at state 4. It ends with a final prestate which generates a state for the hypothetical base. At this state the original confluence for ∂DE is satisfied. Therefore this state is also a state of the original system, namely state

system base	comment	PD	$\Box DE$	∂PD	∂DE	∂IN	activity
original	state 4	H	$\{-,0,+\}$	$+$	$+$	$+$	
hypothetical	$[\partial DE \to -]$	H	$\{-,0,+\}$	$++$	$++$	$++$	
			$\{-,0,+\}$F				1
						$++$F	1
					$--$F		1
				$--$F			1
	transition result	H	$\{-,0,+\}$	$-$	$-$	$+$	
original	state 6						

TABLE 21. The switch of ∂DE at state 4 of the model of Table 4

6. The tendency switch $[\partial DE \to -]$ at state 4 is feasible and leads to state 6 as the transition result.

In the hypothetical base all directionals are anchored. Therefore only activity 1 is applied in the readjustment process realization shown by Table 21.

4.10.4. The tendency switch of ∂AA at state 1 of system A. Table 22 shows the consequences of a tendency switch of ∂AA from $-$ to $+$ at state 1 of system A. In the hypothetical base for $[\partial AA \to +]$ a transition result is reached which is not a state of system A. This switch is not feasible. Therefore the halfway switch $[\partial AA \to 0]$ is examined. The transition result reached in the hypothetical system for the halfway switch fails to be a state of the original system. Therefore the switch $[\partial AA \to +]$ is neither feasible, nor semifeasible but infeasible.

The only activity used in Table 22 is activity 1. Though system A is not anchored, the two hypothetical systems are anchored.

4.10.5. The tendency switch of ∂BA at state 1 of system B. Table 23 shows realizations of the readjustment process in the hypothetical bases for $[\partial BA \to -]$ and the halfway switch $[\partial BA \to 0]$. The tendency switch $[\partial BA \to -]$ is not feasible, but it turns out to be semifeasible. State 2 is the new state reached if $[\partial BA \to -]$ becomes effective.

As in the example of 4.10.4 only activity 1 is used since the two hypothetical bases are anchored, even though system B is not anchored.

<table>
<tr><th>system base</th><th>comment</th><th>∂AA</th><th>∂AB</th><th>activity</th></tr>
<tr><td>original</td><td>state 1</td><td>$-$</td><td>$+$</td><td></td></tr>
<tr><td rowspan="2">hypothetical</td><td>$[\partial AA \to +]$</td><td>$--$
$++$F</td><td>$++$

$--$F</td><td>
1
1</td></tr>
<tr><td>transition result</td><td rowspan="2">$+$</td><td rowspan="2">$-$</td><td rowspan="2"></td></tr>
<tr><td>original</td><td>not a state of the original system</td></tr>
<tr><td>original</td><td>state 1</td><td>$-$</td><td>$+$</td><td></td></tr>
<tr><td rowspan="2">hypothetical for the halfway switch</td><td>$[\partial AA \to 0]$</td><td>$--$
00F</td><td>$++$

00F</td><td>
1
1</td></tr>
<tr><td>transition result</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2"></td></tr>
<tr><td>original</td><td>not a state of the original system</td></tr>
</table>

TABLE 22. Infeasibility of the switch $[\partial AA \to +]$ in system A

system base	comment	$\Box BB$	$\Box BC$	∂BA	∂BB	∂BC	activity
original	state 1	$\{0,+\}$	$\{-,0\}$	$+$	$+$	$-$	
hypothetical	$[\partial BA \to -]$	$\{0,+\}$	$\{-,0\}$	$++$	$++$	$--$	
		$\{0,+\}$F					1
			$\{-,0\}$F				1
				$--$F			1
					00F		1
						00F	1
	transition result	$\{0,+\}$	$\{-,0\}$	$-$	0	0	
original	not a state						

system base	comment	$\Box BB$	$\Box BC$	∂BA	∂BB	∂BC	activity
original	state 1	$\{0,+\}$	$\{-,0\}$	$+$	$+$	$-$	
hypothetical for the halfway switch	$[\partial BA \to 0]$	$\{0,+\}$	$\{-,0\}$	$++$	$++$	$--$	
		$\{0,+\}$F					1
			$\{-,0\}$F				1
				00F			1
					00F		1
						00F	1
	transition result	$\{0,+\}$	$\{-,0\}$	0	0	0	
original	state 2						

TABLE 23. Semifeasibility of $[\partial BA \to -]$ at state 1 of system B

CHAPTER 5

Permissibility and stability

5.1. Informal preliminary remarks

In Section 3.10 the notion of a permissible path has been introduced. The definition was based on the concept of the tentative transition diagram which shows all main transitions. However, the readjustment process had not yet been explained in chapter 3. Nevertheless, it was necessary to speak about transitions and transition diagrams in order to explain the reasons for the introduction of the priority ranking as a part of a qualitative dynamic system.

Only now, after the definition and investigation of the readjustment process it has become clear how the tentative transition diagram of a qualitative dynamic system is determined. It is now possible to attack the question whether in the tentative transition diagram of a qualitative dynamic system a permissible path starting with a given state always exists, or in other words, whether the tentative transition diagram of a qualitative dynamic system is always well structured. Theorem 5 will establish the fact that this is the case.

Another purpose of this chapter is the introduction of a definition of stability. Roughly speaking, stability of a stationary state requires a return after at most one tardy transition in the original system. Any number of immediate transitions may happen in the auxiliary base and after the return in the original system. The stability requirement must be satisfied for every expected perturbance.

The question of stability has been discussed heuristically for stationary states of particular examples. It has been pointed out in 2.2 that the state 2 of the model for Hume's specie-flow mechanism should be considered to be stable by any reasonable definition of the term. This example shows that one main transition in the original system must be permitted on the way back to the stable state. In 3.3 the example of a positive perturbance of ∂DE at the stationary state 9 of the simple business cycle model has been discussed heuristically.

The definition of stability in a qualitative dynamic system is a somewhat difficult problem. In quantitative systems definitions of stability involve arbitrarily small ϵ-neighborhoods. It is not possible to mimick such definitions in the framework of qualitative dynamic systems. The theory proposed here takes a different approach.

A perturbance is interpreted as an exogenous influence of short duration. This exogenous influence is added to the main term of the confluence for the perturbed tendency. Thereby the original base is changed to an auxiliary base. One has to look at all "perturbance histories". A perturbance history begins with a sequence of immediate transitions in the auxiliary base, continued until a lasting state for this base is reached. Then the perturbance history returns to the original system. There a further sequence of immediate transitions may follow and then a tardy transition and finally again a sequence of immediate transitions until a lasting state of the original system is reached. This state is the "decisive" one. Stability requires that this decisive state is always the original stationary state for all perturbance histories initiated by an expected perturbance.

Even if in principle a huge number of perturbance histories may have to be examined this does not seem to be the case in particular examples. In order to show instability it is sufficient to find one perturbance history which does not lead back to the stationary state. In the case of stability perturbance histories usually are quite short and not too many of them need to be examined. At least this is true for the examples presented in this book.

Theorem 4 will show that every immediate tendency switch is feasible. It will be necessary to prove five lemmas before theorem 4 emerges as the final conclusion of 5.3. The fact that an immediate tendency switch is always feasible is important for the theory proposed here. Imagine a state at which some infeasible immediate tendency switches are pending, but no other immediate transition causes. Such a state could hardly be called "fleeting".

Another problem investigated in this chapter concerns the possibility that at one state several main transition causes are pending which lead to the same transition result. One may say that for a given state the relationship between a transition cause pending at it and the transition result which it leads to, does not always have an "inverse". In this sense we speak of the "inverse transition problem". The inverse transition problem is not really important for the development of the theory proposed here. It is, however, of some interest, that an immediate transition cause leading from one state to another is uniquely determined by these two states. This is a consequence of lemma 21 which will be proven in 5.4.

Theorem 5 will exclude infinite tentative paths involving immediate transitions only. This result is used in order to prove theorem 6 which shows that a tentative transition diagram is always well structured in the sense that at every state a permissible path starting with this state can be found. However, theorem 5 is not only important for permissibility but also for the definition of stability. If there could be infinite sequences of immediate transitions, then a return from the auxiliary base to the original system would not be guaranteed.

5.2. Readjustment results and transition results

Let $p_0, \dots, p_N$ be a realization of the readjustment process in the system $\Phi = (\Lambda, \Gamma, \rho, \alpha)$ beginning with a start p_0. In view of theorem 3 in 4.8 the final prestate p_N is uniquely determined by p_0. We use the notation $h(p_0)$ for this final prestate. The prestate $h(p_0)$ is called the **readjustment result** of p_0 in Φ. Let $p_0(\omega, s)$ be the transition start for a shift or lag extinction ω pending at a state s. In this case we also write $h(\omega, s)$ instead of $h(p_0(\omega, s))$ and we refer to $h(\omega, s)$ as the **readjustment result** of ω at s.

Now assume that $\omega = [\partial XY \to d]$ is a tendency switch pending at a state s. In order to determine whether ω is feasible at s one has to look at a realization $p_0, \dots, p_N$ of the readjustment process in the hypothetical base $B_\omega = (\Lambda, \Gamma_\omega)$, beginning with the transition start $p_0 = p_0(s)$ (see 3.2 and 3.4). The final prestate p_N is saturated in B_ω but not necessarily in Φ. The tendency switch ω is **feasible** if and only if p_N is saturated in Φ. If this is the case, then $h(\omega, s)$ denotes the final prestate p_N and $h(\omega, s)$ is called the **readjustment result** of ω at s.

Suppose that ω is a tendency switch from $-$ to $+$ or from $+$ to $-$ and that ω is not feasible. Then we have to look at the halfway switch $\mu = [\partial XY \to 0]$. Let $p'_0, \dots, p'_N$ be a realization of the readjustment process in the hypothetical base B_μ, beginning with $p'_0 = p_0(s)$. The final prestate p'_N is saturated in B_μ but not necessarily in Φ. The tendency switch ω is **semifeasible** at s, if and only if ω is not feasible and p'_N is saturated in Φ. If this is the case, then $h(\omega, s)$ denotes the final prestate p'_N and is called the readjustment result of ω at s.

A main transition cause is called **realizable** at s, if it is a shift, a lag extinction, or a feasible or semifeasible tendency switch pending at a state s. All main transition causes with the exception of infeasible tendency switches are realizable.

We have defined a readjustment result $h(\omega, s)$ for every realizable main transition cause ω pending at a state s. We refer to the function h as the **readjustment result function**. The readjustment result $h(\omega, s)$ is always a saturated prestate for Φ.

In 4.3 the notation $g(p)$ has been introduced for the state generated by a saturated prestate. We now introduce the notation

$$z(\omega, s) = g(h(\omega, s))$$

The state $z(\omega, s)$ is called the **transition result** of ω at s and the function z is the **transition result function**. Obviously z is defined for every realizable main transition cause at a state s.

5.3. The feasibility of immediate tendency switches

In the derivation of the results of this chapter it will be convenient to make use of the notion of the anchorage level of an anchored directional. The anchorage level is recursively defined as follows:

(i) The **anchorage level** of a directional is 1, if there are no other directionals on the right hand side of its confluence or restriction equation.
(ii) For $k = 2, 3, \dots$ the **anchorage level** of an anchored directional is k if on the right hand side of its confluence or restriction equation all directionals have anchorage levels $1, \dots, k-1$ and at least one of these directionals has the anchorage level $k-1$.

In the hypothetical base $B_\omega = (\Lambda, \Gamma_\omega)$ for a tendency switch $\omega = [\partial XY \to d]$ pending at a state s, the tendency ∂XY is anchored and has anchorage level 1, even if it is not anchored in the original system Φ. Therefore it is important to distinguish between anchorage levels in Φ and B_ω. However, it can be seen immediately that every directional which is anchored in Φ is also anchored in B_ω.

In the following we construct an **anchorage realization** $p_0, \dots, p_N$ in B_ω for every tendency switch $\omega = [\partial XY \to d]$. This realization begins with the transition start $p_0 = p_0(s)$, for ω at s and is continued as follows: First all directionals with anchorage level 1 in Φ are adapted and confirmed, then those with anchorage level 2 in Φ, and so on. If ∂XY is anchored in Φ with anchorage level k, then the realization is chosen in such a way that ∂XY is adapted and confirmed as the last one among all directionals of anchorage level k in Φ. If ∂XY is not anchored in Φ, then ∂XY is adapted and confirmed immediately after all directionals anchored in Φ. It is clear that a realization with these properties can be constructed.

A tendency switch of **anchorage level** k is a tendency switch of an anchored tendency with **anchorage level** k. If this tendency switch is immediate we speak of an **immediate** tendency switch of **anchorage level** k.

LEMMA 16. *Let ∂XY be an anchored tendency with anchorage level k in Φ and let $\omega = [\partial XY \to d]$ be a tendency switch pending at a state s in Φ. Moreover let $p_0, \dots, p_N$ be an anchorage realization for ω and let ∂XY be adapted and confirmed in the step from p_m to p_{m+1}. Then the following statements hold:*

1. *At p_m all system specific restrictions adapted and confirmed in the steps from p_0 to p_m are firm and have the same values as at s.*
2. *At p_m all tendencies different from ∂XY and anchored in Φ with anchorage levels of at most k are firm and for each of them the common value of its left and right tendencies is its value at s.*
3. *The tendency switch ω is feasible at s.*

4. *If ω is an immediate tendency switch then the number of immediate tendency switches of anchorage level k in Φ pending at the transition result $z(\omega, s)$ is smaller than the number of immediate tendency switches of anchorage level k in Φ pending at s.*

PROOF. The anchorage realization begins with $p_0 = p_0(s)$. The hypothetical base B_ω and the base B of Φ differ only with respect to the confluence for ∂XY. The tendency ∂XY does not appear on the right hand side of confluences or restriction equations adapted and confirmed in the steps from p_0 to p_m. At $p_0 = p_0(s)$ all these directionals are adjusted in B and therefore also in B_ω. It follows that in no step from p_0 to p_m the value of a system specific restriction or a left or right tendency is changed. Therefore the first two assertions of the lemma hold.

It is a consequence of the first two assertions that at p_m all pieces on the right hand side of the confluence for ∂XY have the same values as at s. Therefore ∂XY is adjusted at p_m not only in B_ω but also in Φ. We can conclude that ω is feasible at s. The third statement holds.

As we have seen above all directionals anchored in Φ with an anchorage level of at most k are adjusted and firm at p_{m+1}. Let ∂VW be a tendency anchored in Φ with anchorage level k and assume that ∂VW is different from ∂XY. At p_m and therefore also at p_N not only ∂VW has the same value as at s but also the right hand side of the confluence for ∂VW. It follows that an immediate tendency switch of ∂VW is pending at the transition result $z(\omega, s)$ if and only if it is also pending at s. However, an immediate tendency switch of ∂XY is not pending at $z(\omega, s)$. It follows that the number of immediate tendency switches of anchorage level k is smaller at $z(\omega, s)$ than at s. Therefore the fourth statement holds. This completes the proof of the lemma. ☐

LEMMA 17. *Let ∂XY be a tendency which is not anchored in Φ and let $\omega = [\partial XY \to d]$ be a tendency switch pending at a state s. Moreover let $p_0, \dots, p_N$ be an anchorage realization for ω and let p_m be the first prestate at which all directionals anchored in Φ are firm. Then the following statements hold:*

1. *At p_m all system specific restrictions are firm and have the same value as at s.*
2. *At p_m all tendencies which are anchored in Φ are firm and the left and right tendency of each of them has the value of the tendency at s.*
3. *At p_m all tendencies which are not anchored in Φ are loose and the left and right tendency of each of them has the value of the tendency at s. Moreover at p_m all of them with the exception of ∂XY are adjusted in B_ω.*

PROOF. According to the definition of an anchorage realization all directionals anchored in Φ and no others are adapted and confirmed in the steps from p_0 to p_m. Nothing else than the confluence for ∂XY is different in the hypothetical base B_ω and the base B of Φ. At $p_0 = p_0(s)$ all directionals are univalued and adjusted in B. With the exception of ∂XY they are also adjusted in B_ω. The adaptation and confirmation of directionals anchored in Φ therefore does not change the values of system specific restrictions. Consequently the first statement holds. The same is true for the left and right tendencies of variables whose tendencies are anchored in Φ. Therefore the second statement holds.

Obviously the left and right tendencies of variables with tendencies not anchored in Φ are not changed in the steps from p_0 to p_m. With the exception of ∂XY the confluences of these tendencies in B_ω are satisfied at s and therefore also at $p_0(s)$ and p_m. Consequently, the third statement holds. This completes the proof of the lemma. □

LEMMA 18. *Under the assumptions of lemma 17 the tendency ∂XY is adapted and confirmed in the step from p_m to p_{m+1}. In the steps from p_{m+1} to p_N activities are applied to tendencies not anchored in Φ and different from ∂XY and no other directionals. Moreover the following two statements hold for $k = m+1, \ldots, N-1$:*

1. *If ∂VW is an adjusted non-zero tendency at p_k in B_ω and an activity is applied to ∂VW in the step from p_k to p_{k+1} then this activity is either activity 1 or activity 4.*
2. *If ∂VW is an adjusted non-zero tendency at p_k in B_ω to which no activity is applied in the step from p_k to p_{k+1} and if the right tendency of a tendency ∂TU in the main term of the confluence for ∂VW changes its value from $\partial TU_R = 0$ to $\partial TU_R = c$ with $c \neq 0$ in the step from p_k to p_{k+1} then ∂VW is an adjusted non-zero tendency at p_{k+1} in B_ω.*

PROOF. The assertions of lemma 18 before the two statements 1. and 2. are immediate consequences of the definition of an anchorage realization and of lemma 17. The first statement follows by the fact that an adjusted non-zero tendency is not of the required type for activities 2, 3, and 5. It remains to prove the second statement.

First consider the case that the confluence for ∂VW has the form

$$\partial VW = T$$

If ∂TU_R changes its value from 0 to c in the step from p_k to p_{k+1} then activity 1 or 4 is applied to ∂TU and the values of the right tendencies of all variables other than TU remain unchanged. Let T_0 and T_1 be the values of T at p_k and p_{k+1}, respectively and let b be the value of ∂VW_L at p_k. Since ∂VW is a non-zero tendency at p_k we have $b \neq 0$. The tendency ∂VW is adjusted at p_k. Therefore

$b \in T_0$ holds. Since T is a direction sum either $T_0 = \{b\}$ or $T_0 = \{-, 0, +\}$ is true. This yields

$$T_1 = T_0 + c = \begin{cases} b & \text{for } T_0 = b \text{ and } c = b \\ \{-, 0, +\} & \text{else} \end{cases}$$

Consequently b is an element of T_1. It follows that ∂VW is an adjusted univalued non-zero tendency at p_{k+1}.

We now look at the case that the confluence for ∂VW has the form

$$\partial VW = T \ @ \ R$$

In view of the first statement of lemma 17 the restriction R does not change its value in the step from p_k to p_{k+1}. The proof for $\partial VW = T$ works without any essential change for the case $R = \{-, 0, +\}$. Therefore in the following we assume that R has at most two elements. We shall show that in this case we either have

$$R = \{b\}$$

or

$$b \in T_1 \cap R$$

It is clear that b must be in R. Otherwise ∂VW would not be adjusted at p_k. Therefore $R = \{b\}$ holds if R has only one element.

Suppose that R has two elements. Then we have $R = \{0, b\}$, in view of $b \in R$, since R is a convex direction set. The result for T_1 obtained for $\partial VW = T$ remains valid in the presence of a restriction of ∂VW. It follows that b is in the intersection of T_1 and R. Therefore either b is in the intersection of T_1 and R or we have $R = \{b\}$. In both cases b is in the value of the right hand side of the confluence for ∂VW at p_{k+1}. It follows that ∂VW is an adjusted non-zero tendency at p_{k+1} in B_ω. This completes the proof of the lemma. □

LEMMA 19. *Under the assumptions of lemma 17 let $\omega = [\partial XY \rightarrow d]$ be an immediate tendency switch. Then the following two statements hold:*

1. *For $k = m, \ldots, N$ there are no maladjusted non-zero tendencies at p_k in B_ω.*
2. *For $k = m+1, \ldots, N$ a tendency ∂VW which is an adjusted non-zero tendency at p_{k-1} in B_ω is an adjusted non-zero tendency in B_ω at the prestate p_k and the value of ∂VW_L does not change in the step from p_{k-1} to p_k.*

PROOF. We first show that the first statement holds for $k = m$. In view of the second and third statement of lemma 17 there are no maladjusted tendencies at p_m with the exception of ∂XY. However ∂XY is a zero tendency at p_m. Therefore the first statement of lemma 19 holds for $k = m$.

In the step from p_m to p_{m+1} the tendency ∂XY is adapted and confirmed. ∂XY_R changes from zero to d in this step. In view of the second statement of lemma 18 all adjusted non-zero tendencies at p_m are not only adjusted in B_ω at p_m but also at p_{m+1}. There is only one non-zero tendency at p_{m+1} which is not an adjusted non-zero tendency at p_m, namely ∂XY. The tendency ∂XY is also adjusted at p_{m+1}. Therefore the two statements of lemma 19 are valid for $k = m + 1$.

With this result as an induction start, we now prove by induction that the two statements of lemma 19 hold. We show for $k = m + 1, \dots, N - 1$ that the two statements of lemma 19 hold for $k + 1$ if they hold for k. Suppose that they are valid for k. Consider a non-zero tendency ∂VW at p_k. Since the first statement holds for k it follows that ∂VW is adjusted at p_k in B_ω. Suppose that an activity is applied to ∂VW in the step from p_k to p_{k+1}. Then this activity must be activity 1 or 4. Therefore ∂VW is adjusted and firm at p_{k+1}. Moreover, these activites do not change ∂VW_L in the step from p_k to p_{k+1}. Therefore in this case the second statement holds for $k + 1$.

Now assume that no activity is applied to ∂VW in the step from p_k to p_{k+1}. Then an activity is applied to another tendency ∂TU. This can be one of the activities 1, 3, 4, and 5, but not 2 since there are no maladjusted non-zero tendencies at p_k in B_ω. For the same reason activities 1 or 4 cannot change the value of ∂TU_R from a non-zero direction to zero if ∂TU is a non-zero tendency at p_k. Activity 3 changes left values only. Activity 5 confirms loose adjusted zero tendencies. It follows by lemma 3 in 4.6 that a zero tendency must be univalued, since for split tendencies the right tendency is zero. Therefore confirmation of loose adjusted zero tendencies changes neither left nor right tendencies. It follows that the application of an activity to another tendency ∂TU may change the value of ∂TU_R from zero to a value different from zero but never from a non-zero direction to zero. The second statement of lemma 18 permits the conclusion that ∂VW remains an adjusted non-zero tendency at p_{k+1} in B_ω if no activity is applied to ∂VW in the step from p_k to p_{k+1}. Obviously in this case, the value of ∂VW_L is not changed in the step from p_k to p_{k+1}.

We have seen that the second statement holds for $k + 1$ if the two statements hold for k. It remains to show that under this assumption the first statement is valid for $k + 1$. Suppose that there is a non-zero tendency ∂RS at p_{k+1} which is maladjusted in B_ω. Then at p_k this tendency ∂RS cannot be adjusted in B_ω, since the second statement of lemma 19 holds for $k + 1$ and it cannot be maladjusted in B_ω in view of the validity of the first statement for k. This is a contradiction. It follows that the two statements of lemma 19 hold for $k+1$. This completes the proof of the lemma. □

LEMMA 20. *Under the assumptions of lemma 17 let* $\omega = [\partial XY \to d]$ *be an immediate tendency switch. Then the following statements hold:*

1. *Let* c *with* $c \neq 0$ *be the value of a tendency* ∂RS *at* s*. Then*
$$\partial RS_L = \partial RS_R = c$$
holds at p_N*.*
2. *The immediate tendency switch* $\omega = [\partial XY \to d]$ *is feasible at* s
3. *At* $z(\omega, s)$ *the number of tendencies with values different from zero is greater than at* s
4. *An immediate tendency switch* $\omega_1 = [\partial PQ \to d_1]$ *of anchorage level* k *in* Φ *is pending at* $z(\omega, s)$ *if and only if it is pending at* s*.*

PROOF. It follows by the second and third statement of lemma 17 that at p_m the tendency ∂RS in the first statement of lemma 20 is adjusted in B_ω and that there ∂RS_L and ∂RS_R are equal to c. With the help of an easy induction argument the second statement of lemma 19 yields the conclusion that not only at p_m but also at $p_{m+1}, \ldots, p_N$ the left and right tendencies of ∂RS are equal to c. The first statement of this lemma is true.

Consider the confluence for ∂XY in Φ. The value of the right hand side of this confluence at s must be $\{-, 0, +\}$, since otherwise no immediate switch of ∂XY could be pending at s (see Table 9 in 3.1). A boundary restriction or a system specific restriction R – if there is any – must have the value $\{-, 0, +\}$ at s and the same is true for the main term T of the confluence for ∂XY.

In view of the first statement of lemma 17 the restriction R – if there is any – has the value $\{-, 0, +\}$ at p_m and therefore also at $p_{m+1}, \ldots, p_N$. The immediate tendency switch $\omega = [\partial XY \to d]$ is feasible at s if at p_N the original confluence for ∂XY is satisfied. This is the case, if the main term T has the value $\{-, 0, +\}$ at p_N.

For the combination of values of scaled variables at s the main term T may depend on the values of current tendencies. In view of the first statement of this lemma, a tendency with a value different from zero at s has left and right tendencies with this value at p_N. Let T_0 and T_1 be the value of T at s and p_N, respectively. At p_N some tendencies with zero values at s may be non-zero tendencies. Let D be the sum of these "new" non-zero tendencies at p_N. We have:

$$T_1 = T_0 + D$$

Since T_0 is $\{-, 0, +\}$ the value of T_1 is $\{-, 0, +\}$ regardless of what is the value of D, since $-$ and $+$ are components of T_0 and therefore of T_1, too (see 2.4). It follows that ω is feasible at s. The second statement of the lemma is true.

All tendencies with values different from zero at s are non-zero tendencies at p_N. However at p_N there is at least one additional non-zero tendency, namely ∂XY with the value zero at s. The number of non-zero tendencies at p_N is the number of tendencies different from zero at the transition result $z(\omega, s)$. Therefore the third statement of the lemma is true.

Consider a tendency ∂PQ, anchored in Q with the anchorage level k. In view of the second statement of lemma 17 the tendency ∂PQ is firm at p_m and therefore also at $p_{m+1}, \dots, p_N$. Moreover at these prestates ∂PQ_L and ∂PQ_R have the same value as ∂PQ at s. It follows that the value of ∂PQ at $z(\omega, s)$ is the same one as at s. All pieces on the right hand side of the confluence for ∂PQ are anchored in Φ. The argument about ∂PQ can also be applied to the current tendencies among these pieces. Therefore at $z(\omega, s)$ not only ∂PQ has the same value as at s but also the right hand side of the confluence for ∂PQ. It follows that an immediate tendency switch $\omega_1 = [\partial PQ \to d_1]$ is pending at $z(\omega, s)$ if and only if it is pending at s. The fourth statement of this lemma is true. This completes the proof of the lemma. □

THEOREM 4. *Every immediate tendency switch pending at a state s is feasible.*

PROOF. The third statement of lemma 16 shows that the assertion is true if ∂XY is anchored in Φ. The first statement of lemma 20 permits the same conclusion for the case that ∂XY is not anchored in Φ. □

5.4. The inverse transition problem

Let $\Omega(s)$ be the set of all main transition causes pending at a state s and let $Z(s)$ be the set of all states s' with

$$s' = z(\omega, s)$$

for an $\omega \in \Omega(s)$. For a fixed s we may look at the restriction of the transition result function z to $\Omega(s)$ as a mapping from $\Omega(s)$ onto $Z(s)$. The question arises whether this mapping has an **inverse** $z^{-1}(s', s)$ which assigns a unique main transition cause $\omega \in \Omega(s)$ with $s' = z(\omega, s)$ to every $s' \in Z(s)$. We call this the **inverse transition problem**. As we shall see the answer to the question is no.

Let s be a fleeting state and let $\Omega_0(s)$ be the set of all immediate transition causes ω pending at s. Moreover let $Z_0(s)$ be the set of all $s' = z(\omega, s)$ for an $\omega \in \Omega_0(s)$. For a fixed fleeting state s we may look at the restriction of z to $\Omega_0(s)$ as a mapping from $\Omega_0(s)$ onto $Z_0(s)$. Again the question arises whether this mapping has an inverse $z^{-1}(s', s)$ which assigns a uniquely determined immediate transition cause $\omega \in \Omega_0(s)$ to every $s' \in Z_0(s)$. We call this the **inverse immediate transition problem**. As we shall see the answer to this question is yes.

Even if the inverse transition problem is not important for the theory proposed here, the positive answer to the inverse immediate transition problem is of some minor interest for the development of our formalism. If s is a fleeting state and s' is a state in $Z_0(s)$ then we can speak of **the** immediate transition cause ω leading from s to $s' = z(\omega, s)$. This facilitates some of the definitions.

Lemma 21 will show that the non-uniqueness of a transition cause ω with $s' = z(\omega, s)$ is a phenomenon of very limited scope. If for a given pair of states s and s' with $s' \in Z(s)$ two transition causes exist which lead from s to the transition result s', then each of the two transition causes must be a feasible non-anchored tardy tendency switch. It is not excluded that in the tentative transition diagram two nodes are connected by several links, but if this happens these multiple links all represent feasible tardy tendency switches. Moreover these switches are not anchored in Φ.

LEMMA 21. *Let ω_1 and ω_2 be two different realizable main transition causes pending at the same state s. If we have*

$$z(\omega_1, s) = z(\omega_2, s) = s'$$

then ω_1 and ω_2 are feasible tardy tendency switches at s which are not anchored in Φ.

PROOF. We first show that ω_1 or ω_2 cannot be a shift or a lag extinction. Such transitions change values of scaled variables or lagged tendencies which remain fixed in the readjustment process. If ω_1 and ω_2 are shifts or lag extinctions then different components of s are changed and it is not possible that the same transition result is reached. If ω_1 is a shift or lag extinction and ω_2 is a feasible tendency switch then one component is changed by ω_1 but in $z(\omega_2, s)$ this component has the same value as at s. We can conclude that $z(\omega_1, s)$ and $z(\omega_2, s)$ are different.

It remains to look at the case that ω_1 and ω_2 are tendency switches. We can exclude the subcase in which ω_1 and ω_2 are switches of the same tendency ∂XY. In this case ω_1 and ω_2 must be immediate switches in opposite directions which by Theorem 4 lead to different transition results (see Table 9).

In the following it will be assumed that ∂XY and ∂VW are different tendencies and that ω_1 and ω_2 have the forms

$$\begin{aligned} \omega_1 &= [\partial XY \to d_1] \\ \omega_2 &= [\partial VW \to d_2] \end{aligned}$$

We first show that $z(\omega_1, s)$ and $z(\omega_2, s)$ are different if ω_1 and ω_2 are anchored in Φ. Let k_1 and k_2 be the anchorage levels of ω_1 and ω_2, respectively. Without loss of generality we can assume $k_1 \leq k_2$. Let $p_0, \ldots, p_N$ be an anchorage realization for ω_2 at s. It follows by the first two statements of lemma 16 that ω_1 is still pending at $z(\omega_2, s)$. Therefore $z(\omega_1, s)$ and $z(\omega_2, s)$ are different.

Now assume that ω_1 is anchored in Φ with anchorage level k, and ω_2 is not anchored in Φ. As before let $p_0, \ldots, p_N$ be an anchorage realization for ω_2 at s and let p_m be the first prestate in $p_0, \ldots, p_N$ at which all anchored directionals are firm. It follows by lemma 17 that at p_m the tendency ∂XY is firm and the right hand side of its confluence has the same value as at s. Consequently ω_1 is pending at $z(\omega_2, s)$ but not at $z(\omega_1, s)$. The two transition results are different.

In the following we assume that ω_1 and ω_2 are not anchored in Φ and that ω_1 is an immediate tendency switch. This is the only remaining case. Let $p_0, \ldots, p_N$ be an anchorage realization for ω_1 and let p_m be the first prestate at which all anchored tendencies are firm. The tendency ∂XY is adapted and confirmed in the step from p_m to p_{m+1} and in this step the common value of ∂XY_L and ∂XY_R changes from zero to d_1.

Since ∂XY does not appear on the right hand side of confluences and restriction equations for directionals anchored in Φ, the values of these directionals at $p_0, \ldots, p_N$ are identical to their values at s. Moreover every tendency ∂TU which is anchored in Φ is univalued at $p_0, \ldots, p_N$.

Let ∂RS be a tendency different from ∂XY and not anchored in Φ. Let c be the value of ∂RS at s. In the steps from $p_0, \ldots, p_m$ the tendency ∂RS is univalued and $\partial RS_L = \partial RS_R = c$ holds. For $c \neq 0$ it follows by the first statement of lemma 20 that $\partial RS_L = \partial RS_R = c$ is valid at p_N. Consequently ω_2 is pending at $z(\omega_1, s)$ if the value of ∂VW at s is different from zero. In the following we assume that the value of ∂VW at s is zero.

By what has been said about directionals anchored in Φ, at $p_0, \ldots, p_N$ the restriction of ∂VW – if there is one – has its value at s. At s the main term of the confluence for ∂VW has the value $\{-, 0, +\}$, since ω_2 is a tendency switch (see 3.1). In this main term tendencies may appear as components whose values are zero at s. If one of these tendencies changes its value from zero to $-$ or $+$, the value $\{-, 0, +\}$ of the main term remains unchanged. In view of lemma 19 this has the consequence that at $p_0, \ldots, p_N$ the main term of the confluence for ∂VW has the value $\{-, 0, +\}$ at $p_0, \ldots, p_N$. We can conclude that at $p_0, \ldots, p_N$ the right hand side of the confluence for ∂VW has the same value as at s. Consequently ∂VW remains adjusted at its value zero and is finally confirmed by activity 5. Therefore ω_2 is pending at $z(\omega_1, s)$. The transition results $z(\omega_1, s)$ and $z(\omega_2, s)$ are different. This completes the proof of the lemma. □

Remark. *It follows by lemma 21, that for $s' \in Z_0(s)$ the transition cause ω with $s' = z(\omega, s)$ is uniquely determined by s and s'. If an immediate transition* *cause ω leads from s to s' then no other immediate or tardy transition cause ω' leads from s to s'.*

5.5. The system D

Table 24 shows the example of a very simple qualitative dynamic system with two unscaled variables DA and DB and two states 1 and 2. It can be seen as follows that this system D has no other states. In view of the confluence for ∂DA the value of ∂DA cannot be $-$ at a state unless the value of ∂DB is $-$, too. Therefore state 1 is the only one at which ∂DA has the value $-$. Suppose that ∂DA has the value zero. Then it follows by the confluence for ∂DB that ∂DB has the value $+$. This leads to the conclusion that ∂DA has the value $+$ contrary to the assumption that this value is zero. Therefore there is no state at which the value of ∂DA is zero. Now assume that at a state ∂DA has the value $+$. Then ∂DB must have the value $+$. Obviously state 2 is the only one at which ∂DA has the value $+$.

Variables

DA, DB unscaled

Confluences

$\partial DA = \{+\} + \partial DB$

$\partial DB = \{+\} + \partial DA$

States and priorities

state	∂DA	∂DB	priority rank 1
1	$-$	$-$	$[\partial DA \to +], [\partial DB \to +]$
2	$+$	$+$	/

Table 24. The system D

In system D two feasible tardy tendency switches $[\partial DA \to +]$ and $[\partial DB \to +]$ are pending at state 1. Each of the two switches leads to state 2 as the transition result. The two switches are not anchored in D. Table 25 shows realizations of the readjustment process in the hypothetical bases for these tendency switches.

The example of system D shows that the special case of two different transition causes pending at a state and leading to the same transition result can occur. However, lemma 21 puts narrow limits on the scope for this non-uniqueness of an "inverse".

base	comment	∂DA	∂DB	activity
original	state 1	$-$	$-$	
hypo-thetical	$[\partial DA \to +]$	$--$	$--$	
		++F		1
			++F	1
original	state 2	$+$	$+$	

original	state 1	$-$	$-$	
hypo-thetical	$[\partial DB \to +]$	$--$	$--$	
			++F	1
		++F		1
original	state 2	$+$	$+$	

TABLE 25. Two tardy switches between two states

5.6. The finiteness of immediate transition chains

An **immediate transition chain** is a finite sequence $s_0, \dots, s_M$ or an infinite sequence $s_0, s_1, \dots$ such that the following two conditions are satisfied:

(a) For $m = 1, \dots, M$ in the case of a finite sequence and for $m = 1, 2, \dots$ in the case of an infinite sequence the state s_m is the transition result

$$s_m = z(\omega_{m-1}, s_{m-1})$$

where ω_{m-1} is an immediate transition cause ω_{m-1} pending at s_{m-1}.

(b) In the case of a finite immediate transition chain $s_0, \dots, s_M$ the state s_M is a lasting state (see 3.5).

An **immediate transition loop** is a sequence $s_0, \dots, s_M$ with $s_M = s_0$ such that

$$s_m = z(\omega_{m-1}, s_{m-1})$$

holds for $m = 1, \dots, M$ where ω_{m-1} is an immediate transition cause pending at s_{m-1}.

In view of the remark after lemma 21 it is clear that in these definitions ω_{m-1} is uniquely determined by s_{m-1} and s_m. We refer to this uniquely determined immediate transition cause pending at s_{m-1} as the transition cause between s_{m-1} and s_m. There is a close connection between the concepts of an infinite immediate transition chain and an immediate transition loop. This connection is expressed by the following lemma.

LEMMA 22. *A qualitative dynamic system has an infinite immediate transition chain, if and only if it has an immediate transition loop.*

PROOF. Consider an infinite immediate transition chain $s_0, s_1, \ldots$. Since the number of states is finite, one of the states in the chain, say s_k, must be equal to a later state s_m in the chain. Obviously $s_k, \ldots, s_m$ is an immediate transition loop.

Now consider an immediate transition loop $s_0, \ldots, s_m$. We can construct an infinite immediate transition chain $s'_0, s'_1, \ldots$ by repeating the loop again and again:

$$s'_{kM+m} = s_m \quad \text{for } m = 0, \ldots, M-1 \text{ and } k = 0, 1, \ldots$$

This completes the proof of the lemma. □

THEOREM 5. *Every immediate transition chain is finite.*

PROOF. In view of lemma 22 it is sufficient to show that a qualitative dynamic system has no immediate transition loop. The proof will be indirect. Let $s_0, \ldots s_M$ be an immediate transition loop. Suppose that there is an immediate shift ω_{j-1} of a scaled variable XY between two states s_{j-1} and s_j of the chain. Then the value of XY is changed from a point to a range in the transition from s_{j-1} to s_j. No further immediate transition can change the value back from the range to the point. Therefore an immediate transition loop cannot involve immediate shifts. Every ω_{j-1} with $j = 1, \ldots, M$ must be an immediate tendency switch.

Assume that at least one transition cause between two consecutive states of the loop $s_0, \ldots, s_M$ is an anchored immediate tendency switch. Let k_0 be the lowest anchorage level of an anchored immediate tendency switch between two consecutive states of the loop. Let ω_{j-1} between s_{j-1} and s_j be an anchored immediate tendency switch with anchorage level k_0. For $i = 0, \ldots, M$ let μ_i be the number of anchored immediate tendency switches with anchorage level k_0 pending at s_i. It follows by the fourth statement of lemma 16 that we have $\mu_j < \mu_{j-1}$.

It will now be argued that for every $i = 1, \ldots, M$ with $i \neq j$ we have $\mu_i \leq \mu_{i-1}$. Suppose that the transition cause ω_{i-1} between s_{i-1} and s_i is an anchored immediate switch of a tendency ∂VW with anchorage level k. We have $k \geq k_0$. In view of the first two statements of lemma 16 at p_m, before ∂VW is adapted and confirmed all other tendencies ∂TU with anchorage levels up to k are firm. Moreover their values and those of the right hand sides of their confluences are the same ones as at s. Therefore μ_i is smaller than μ_{i-1} for $k = k_0$ and equal to μ_{i-1} for $k > k_0$.

Now assume that the transition cause ω_{i-1} between s_{i-1} and s_i is an immediate tendency switch which is not anchored. It follows by the fourth statement of lemma 20 that in this case we have $\mu_i = \mu_{i-1}$.

A simple induction argument yields $\mu_{j-1} < \mu_{j-1}$. Consequently an immediate transition loop cannot involve anchored immediate tendency switches with anchorage level k_0 and therefore no anchored immediate tendency switches at all. Every ω_i with $i = 1, \ldots, M$ must be an immediate tendency switch which is not anchored.

Let λ_i be the number of tendencies with values different from zero at s_i. It follows by the third statement of lemma 20 that we have $\lambda_i > \lambda_{i-1}$ for $i = 0, \ldots, M$. Obviously this is a contradiction. We can conclude that an immediate transition loop does not exist. This completes the proof of the theorem. □

5.7. Existence of a permissible path starting at a given state

In this section it will be shown that the tentative transition diagram is well structured or, in other words, that a permissible path can be found starting from any given state.

THEOREM 6. *The tentative transition diagram of a qualitative dynamic system is well structured.*

PROOF. The theorem asserts that a permissible path starting at s can be found for every state s. Let s_1 be a fixed arbitrary state. We first construct a special tentative path starting at s_1 and then show that this path is permissible.

The basic idea of this construction is avoiding unresolved shifts and lag extinctions by making sure that every realizable main transition cause of positive rank pending at a lasting state s is realized again and again, if the constructed path turns out to be infinite. For this purpose these transition causes are lined up in an arbitrary fixed order. Every time s is reached again, the next transition cause in this order is realized.

Let L be the set of all lasting states. For every $s \in L$ let $J(s)$ be the number of realizable main transition causes of positive rank pending at s. For each $s \in L$ with $J(s) > 0$ we attach one and only one of the numbers $1, \ldots, J(s)$ to each realizable main transition cause ω of positive rank pending at s. This number is denoted by $\lambda(\omega, s)$. Let F be the set of all fleeting states. For every $s \in F$ let ω_s be a fixed immediate transition cause of rank 1 at s.

We now construct the special tentative path mentioned above. This path may turn out to be a finite sequence $s_1, \ldots, s_M$ or an infinite sequence $s_1, s_2, \ldots$. The next state s' after a state s on the path is determined by

$$s' = z(\omega_s, s) \text{ for } s \in F$$

Consider the case of a state $s \in L$ on the path. For $J(s) = 0$ the state s is stationary and the construction of the path is not continued beyond s. Let s be reached for the $u-$th time in the $(m-1)$-th episode. Moreover let v_u be the greatest non-negative integer with $u \geq v_u J(s)$ and let ω_u be that realizable main transition cause of positive rank pending at s for which

$$\lambda(\omega_u, s) = u - v_u J(s)$$

holds. The next state s' after s on the path is

$$s' = z(\omega_u, s) \text{ for } s \in L \text{ with } J(s) > 0$$

The tentative path is continued as long as possible. It cannot end with a state s_M unless a stationary state is reached at the M-th episode.

If the tentative path constructed in this way turns out to be finite, then no main transition causes of positive rank are pending at the last state and therefore the tentative path has no unresolved shifts or lag extinctions. In this case the construction yields a permissible path. From now on we assume that the path is infinite.

There is at least one state $s \in L$ which is reached infinitely often by the path $s_1, s_2, \ldots$. This can be seen as follows. Suppose that the m-th episode is the last one in which a state in L is reached. This would mean that $s_{m+1}, s_{m+2}, \ldots$ is an infinite immediate transition chain contrary to theorem 5. Therefore at least one $s \in L$ is reached infinitely often. Let s' be one of these states.

Suppose that ω is an unresolved shift or lag extinction of $s_1, s_2, \ldots$. Since ω is pending at all $s_m, s_{m+1}, \ldots$ from some m on, it must be pending at s'. However, any main transition cause ω pending at s' will again and again give rise to a transition on the path. These transitions lead to a next state at which ω cannot be pending. It follows that there cannot be any s_m such that ω is pending at all episodes of the path from s_m on. Consequently $s_1, s_2, \ldots$ is a permissible path. This completes the proof of the theorem. □

Remark (The rank of a system). *It is a consequence of theorem 6 that the tentative transition diagram always has a rank k^* (see 3.10). From now on we shall refer to this rank k^* as the* **rank of the system**. *Similarly the transition diagram derived from the tentative transition diagram will be called the* **transition diagram of the system**.

The algorithm described in the proof of theorem 6 determines a permissible path starting from any given state but it is not suitable for the computation of the rank of the system. The rank of the system is almost obvious in the simple examples discussed in this book, but for big qualitative dynamic systems this may be quite different. The development of an efficient algorithm for the computation of the rank of a system would be desirable for applications to big systems. However, this question is not further pursued in this book.

5.8. Transitions due to perturbances

The auxiliary base B_ω for a perturbance $\omega = [\partial XY : d]$ with $d = +$ or $d = -$ at a potentially stationary state s of $B = (\Lambda, \Gamma)$ has been introduced in 3.3.1. This auxiliary base is obtained by first replacing the main term T of the confluence for ∂XY by $T + d$ and then applying some simplifying equivalent transformations to $T + d$. The result is a new term T_A which satisfies the conditions (c3), (c4), (c6), and (c7) of 2.8 required for a main term of a confluence (see 3.3.1). Nothing else than the main term of the confluence for ∂XY is different in B_ω and B. In particular, the new main term T_A is accomodated to the same restriction $\triangleright\partial XY$ or $\Box XY$ — if there is any — as the original main term T.

It has been explained that a perturbance $\omega = [\partial XY : d]$ is thought of as a temporary exogenous influence of short duration which adds a component d to the main term of the confluence for ∂XY. As soon as such an influence becomes effective the dynamic process leaves the original system and enters the auxiliary base. The auxiliary base has a very short life time. As soon as the exogenous influence stops to work, the dynamic process returns to the original system. During the short life time of the auxiliary base only immediate transitions can occur. Nevertheless it is assumed that there is sufficient time for an arbitrarily long immediate transition chain. This is based on the idea that immediate transitions are practically instantaneous. In this section a formal description of the consequences of a perturbance will be provided.

An auxiliary base is a base in the sense of the definition in 2.9 (see 3.3.1), but it is not a full fledged qualitative dynamic system. It is not complemented by a priority ranking. All immediate transition causes pending at a state of an auxiliary base are treated as equally plausible. No perturbance assignment is specified for an auxiliary base. Nevertheless transition results in an auxiliary base can be determined with the help of the readjustment process in the same way as in the original system.

The auxiliary base for ω at s has the same list of variables as the original system. However, since the system of confluences and restriction equations is different the states of the auxiliary base are usually not the same ones as those of the original system. Nevertheless the space of prestates is the same one in the original system and the auxiliary base. Moreover, a start in the original system is also a start in the auxiliary base and vice versa.

In order to examine the consequences of a perturbance ω at s one has to examine all possible **reentry histories**. The meaning of this term will now be explained with the help of Table 26. This table shows symbols in the first column, their names in the second one and the base to which they belong in the third one.

Symbols	Names	Base
s	stationary state	original
ω	perturbance at s	
$p_0 = p_0(s)$	perturbance start	auxiliary
$q_0 = h_\omega(p_0)$	readjustment result of p_0	
$a_0 = g(q_0)$	opening state of ω at s	
$a_0, \ldots, a_M$	immediate transition chain	
$q = p_0(a_M)$	return start	original
$p = h(q)$	readjustment result of q	
$e = g(p)$	reentry state	

TABLE 26. Structure of a reentry history

A reentry history begins with a stationary state s of the original base and a perturbance ω at this state. Then a readjustment process in the auxiliary base begins, starting with the prestate $p_0 = p_0(s)$. The prestate $p_0(s)$ is the perturbance start for ω at s (see 4.3). The readjustment result reached from $p_0 = p_0(s)$ in the auxiliary base is denoted by $h_\omega(p_0)$. The prestate $q_0 = h_\omega(s)$ is saturated in the auxiliary base B_ω and therefore generates a state $a_0 = g(q_0)$ of B_ω. We call a_0 the **opening state of** ω **at** s.

The opening state a_0 is the first state in an immediate transition chain $a_0, \ldots, a_M$ for B_ω. If a_0 is lasting, then we have $M = 0$ and a_0 is also the last state of the chain. A transition cause η between two consecutive states of the chain a_{m-1} and a_m may be an immediate shift or an immediate tendency switch. In the case of an immediate switch η the transition from a_{m-1} to a_m involves a hypothetical base of the auxiliary base, denoted by $B_{\omega\eta}$. In view of theorem 4 the immediate tendency switch η is feasible at a_{m-1}. A readjustment process starting with $p_0(a_{m-1})$ in $B_{\omega\eta}$ leads to the readjustment result denoted by $h(a_{m-1}, \eta, \omega)$. This prestate is saturated in B_ω and generates the state a_m of the chain.

The prestate $q = p_0(a_M)$ is called the **return start** of the reentry history. A readjustment process beginning with the return start q in the original system leads to the readjustment result denoted by $h(q)$. The prestate $p = h(q)$ generates the state $e = g(p)$ of the original system. This state e is called the **reentry state**. The reentry history ends with the reentry state.

Different reentry histories may differ with respect to the immediate transition chain $a_0, \ldots, a_M$ in the auxiliary base. The definition of stability will require the examination of all possible reentry histories. The set of all reentry states which can be reached after an expected perturbance $\omega \in \alpha(s)$ at a stationary state s is denoted by $E(\omega, s)$. The union of all $E(\omega, s)$ with $\omega \in \alpha(s)$ is denoted by $E(s)$ and called the **reentry state set** of s. A reentry state $e \in E(s)$ may be reached by more than one reentry history and even after different expected perturbances at s. We refer to a pair (s, e) such that s is a stationary state and e is a reentry state in $E(s)$ as a **perturbance transition** from s to e.

In the following the notion of the extended transition diagram will be explained. The **extended transition diagram** shows all main transitions of the transition diagram and in addition to this each perturbance transition from a stationary state s to a reentry state $e \in E(s)$.

Formally the extended transition diagram is a directed graph with some additional features. The vertices stand for states and the edges represent possible transitions. The edges are either **main edges** associated to main transitions of the transition diagram or **perturbance edges** corresponding to perturbance transitions from stationary states s to reentry states $e \in E(s)$. A value is attached to each main edge, the priority rank of the main transition cause for the represented main transition.

5.9. Definition of stability

The meaning of stability in a qualitative dynamic system is not obvious: There may be different ways in which stability of a stationary state against a perturbance can be defined. It seems to be a minimal requirement that after the perturbance every permissible path starting with a reentry state leads back to the stationary state. However this would mean that one permits very long return paths which may lead far away from the stationary state before they come back to it. The stationary state of an economy would hardly be called stable, if a perturbance creates strong fluctuations lasting for a long time even if the economy eventually comes back to this state. Therefore our theory takes the point of view that not only every permissible path starting with a reentry state should return to the stationary state, but that in addition to this there should be at most one tardy transition on every path of this kind.

The example of the model for Hume's specie-flow mechanism shows that at least one tardy transition must be permitted. However, the question arises why just one and not two or three tardy transitions should be permissible. The answer is that intentionally the most stringent definition is chosen which still seems to be reasonable. One may also look at a stability concept which does not require more than the minimal requirement of a return to the stationary state on every permissible path. The word "recaptor" will be used for a stationary state with this property. The notion of a recaptor can be looked upon as a very liberal stability concept. However we reserve the word "stability" for the concept proposed here, in order to avoid terminological confusion. Nevertheless one can speak about alternative concepts under different names.

We now turn our attention to more formal definitions. A stationary state s is **escapable** by a perturbance ω pending at s, if for at least one reentry state $e \in E(\omega, s)$ a permissible path starting with e exists in the transition diagram such that this path does not lead back to s. A stationary state s is a **recaptor**, if s is not escapable by any expected perturbance $\omega \in \alpha(s)$.

A stationary state s is **destabilizable** by a perturbance ω pending at s, if for at least one reentry state $e \in E(\omega, s)$ a permissible path starting with e exists in the transition diagram, such that this path does not lead back to s after at most one tardy transition. (This does not exclude many immediate transitions on the path.) A stationary state s is **stable** if it is not destabilizable by any expected perturbance $\omega \in \alpha(s)$.

A stationary state s is **unreachable after a perturbance** ω pending at s, if in the transition diagram every permissible path starting with a reentry state $e \in E(\omega, s)$ never comes back to s. A **repulsor** is a stationary state s which is unreachable after every expected perturbance $\omega \in \alpha(s)$.

The notion of a repulsor describes the strongest form of instability one can imagine. A stationary state is **unstable** if it is not stable. An unstable stationary state is not necessarily a repulsor. It may even be a recaptor.

Note that in the definitions above all paths are permissible paths in the transition diagram. There may be more permissible paths in the tentative transition diagram. However, such additional permissible paths are excluded from consideration. The transition diagram is also different from the extended transition diagram. Only main transitions appear in the transition diagram. In the extended transition diagram a path starting at a stationary state s may lead back to s via a perturbance ω' at another stationary state.

5.10. Examples of stability and instability

In this section we will look at questions of stability and instability for the first three examples (not the systems A to D) of qualitative dynamic systems considered up to now.

5.10.1. Hume's specie flow mechanism. This model has been introduced in 2.1 and 2.2. In 3.8.1 the model has been complemented by a priority ranking and a perturbance assignment. The only stationary state is state 2 and the expected perturbances at this state are $[\partial GO : -]$ and $[\partial GO : +]$. In the following we examine the consequences of the perturbance

$$\omega = [\partial GO : +]$$

at state 2. As we have seen in 3.3.1 the confluence for ∂GO in the auxiliary base for this perturbance is as follows:

$$\partial GO = \begin{cases} \{-, 0, +\} & \text{for} \quad TR = D \\ + & \text{for} \quad TR = b \\ + & \text{for} \quad TR = S \end{cases}$$

Table 27 shows the reentry history – there is only one – after the perturbance ω at state 2 and the subsequent return to the stationary state 2 by the tardy transition caused by $[TR \to b]$. The table follows the conventions for readjustment process tables (see 4.10.1 and 4.10.3). The only activity used is activity 1. Therefore the column "activity" is left out.

At the perturbance start TR has the value b and all tendencies are univalued zero tendencies. The right hand side of the confluence for ∂GO in the auxiliary base has the value $+$ for $TR = b$. Therefore activity 1 applied to ∂GO yields $\partial GO_L = \partial GO_R = +$. At the opening state a_0 the immediate shift $[TR \to D]$ is pending. No other immediate transition cause is pending at a_0. Therefore the immediate transition chain continues with the immediate shift of TR from b to D. For $TR = D$ the right hand side of the confluence for ∂GO in the auxiliary base has the value $\{-, 0, +\}$. Activity 1 yields $\partial GO_L = \partial GO_R = +$. The transition result is the state a_1 for the auxiliary base. This state is lasting. Therefore it gives rise to the return start $q = p_0(a_1)$. A realization of the readjustment process beginning with the return start q in the original system leads to state 1 as the reentry state.

The tardy shift $\omega_1 = [TR \to b]$ is the only main transition cause pending at state 1. A realization of the readjustment process in the original system beginning with the transition start for ω_1 at state 1 leads back to the stationary state 2.

There is only one permissible path starting with the uniquely determined reentry state and this path returns to the stationary state 2 after just one tardy transition. It follows that state 2 is not destabilizable by a positive perturbance of ∂GO.

The case of a negative perturbance of ∂GO is analogous. Essentially the same analysis can be applied to this case. The stationary state 2 is not destabilizable by a negative perturbance of ∂GO either. Therefore the stationary state 2 of the model is stable.

As has been explained in 3.8.1 exactly one transition cause is pending at each of the two non-stationary states 1 and 3, namely a tardy shift to state 2. These shifts have priority rank 1. Therefore the transition diagram has rank 1.

Figure 9 shows the extended transition diagram for the model of Hume's specie flow mechanism. State numbers are indicated in the rectangles representing the nodes. On the right of the figure a vertical **scale line** has been drawn. This scale line has a **height interval** for each value of TR, the only scaled variable of the system. If in the figure the height of a rectangle falls into the height interval for a value of TR then TR has this value at the state represented by the rectangle. These **conventions for figures representing extended transition diagrams** will be used in the remainder of this chapter and in chapter 7.

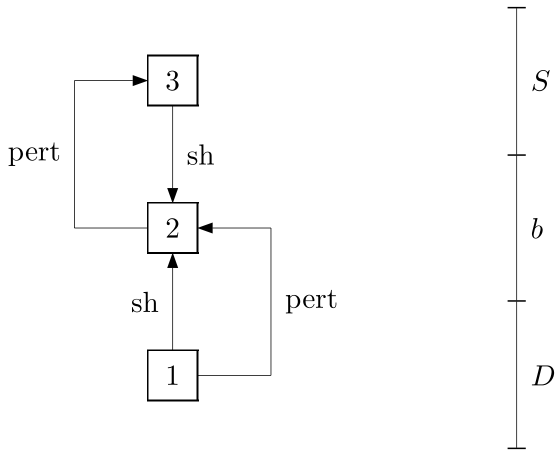


FIGURE 9. The extended transition diagram for Hume's specie flow mechanism

Abbreviations
pert perturbance
sh tardy shift

5.10.2. The simple business cycle model of Table 4. This model has been introduced in 2.5. A priority ranking and a perturbance assignment has been specified by Table 13 in 3.8.2. The only stationary state is state 9. The expected perturbances at state 9 are $[\partial IN : +]$ and $[\partial IN : -]$. In the following we examine the consequences of the perturbance

$$\omega = [\partial IN : +]$$

at state 9. In the auxiliary base B_ω the confluence for ∂IN is as follows:

$$\partial IN = \begin{cases} \{-, 0, +\} & \text{for} \quad PD = b, L \\ + & \text{for} \quad PD = n \\ + & \text{for} \quad PD = H, c \end{cases}$$

The only reentry history for ω at state 9 is shown by Table 28. In the first two steps of each of the three realizations of the readjustment process in this table the anchored directionals $\Box DE$ and ∂IN are adapted and confirmed. In the first realization leading to the opening state a_0 a situation arises in which ∂DE is a maladjusted zero tendency whereas ∂PD is an adjusted zero tendency. Therefore first activity 3 and then activity 4 is applied to ∂DE. Since ∂PD is a zero tendency activity 4 cannot be continued. In the last step ∂PD is mature and can be adapted and confirmed.

At the opening state the immediate shift $[PD \rightarrow L]$ and no other immediate transition cause is pending. This immediate shift has priority rank 1 at a_0. The transition caused by $[PD \rightarrow L]$ leads to the state a_1 of the auxiliary base. Here it is important that at $p_0(a_0)$ the right hand side of the confluence for ∂IN in B_ω has the value $\{-, 0, +\}$. Therefore the equation $\partial IN_L = \partial IN_R = +$ is not changed by the adaptation and confirmation of ∂IN.

The state a_1 is lasting. Therefore a return to the original system follows after a_1. The realization of the readjustment process in the original system beginning with the return start $p_0(a_1)$ leads to state 8, the reentry state, as the transition result. State 8 is a state of the cycle (see Figure 3 in 2.5). Every permissible path starting in the cycle remains in the cycle forever. It follows that state 9 is not stable and even unreachable after ω.

The case of a negative perturbance of ∂IN at state 9 is analogous. Table 29 shows the reentry history for $[\partial IN : -]$. This table is completely parallel to Table 28. State 4, the reentry state is also a state of the cycle. It follows that after an expected perturbance no permissible path leads back to the state 9. Therefore state 9 is not only unstable, but also a repulsor.

In the model of Table 4 all main transitions have rank 1 (see Table 13 in 3.8.2). Therefore the rank of the transition diagram of the model is 1. Figure 10 shows the extended transition diagram. The conventions for figures representing extended transition diagrams explained in 5.10.1 are used.

5.10.3. The model of Table 6. The modified simple business cycle model has been introduced by Table 6 in 2.7. The priority ranking and the perturbance assignment for this model have been specified by Table 14 in 3.8.3.

It has been pointed out in 2.9 and 4.10.2 that all directionals are anchored in the model of Table 6. It can be seen without difficulty that this is also true for all hypothetical and auxiliary bases as well as all hypothetical bases for auxiliary bases. In these modifications directionals may be anchored which are not anchored in the original system, but it is not possible that a directional which is anchored

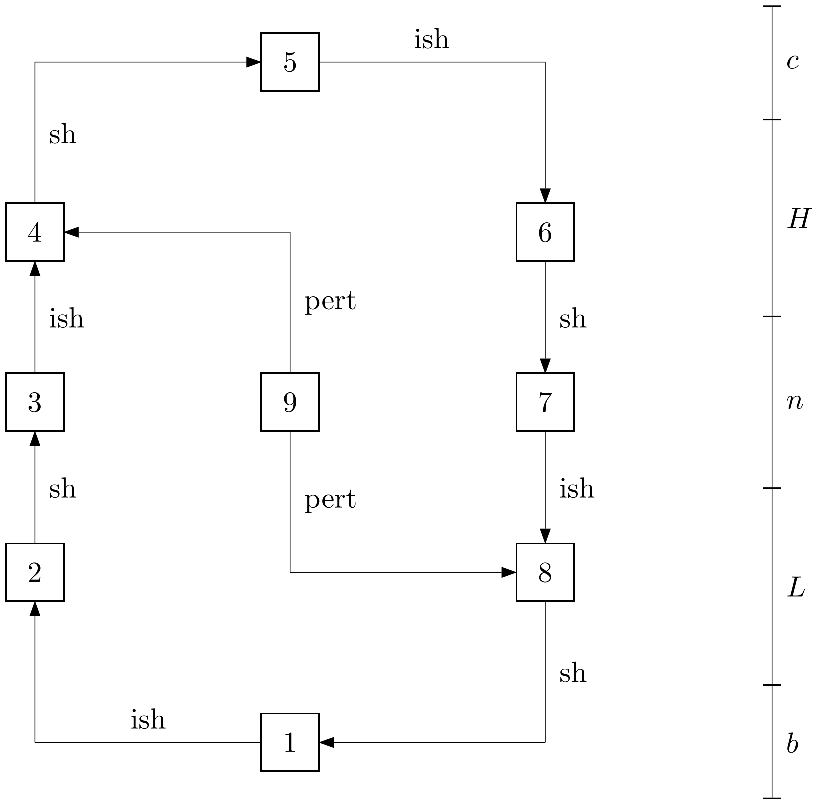


FIGURE 10. The extended transition diagram for the model of Table 4

Abbreviations
ish immediate shift
pert perturbance
sh tardy shift

in the original system loses the property of being anchored in one of these modifications. Therefore the directionals can always be adapted and confirmed in the following fixed order: $\Box DE, \partial IN, \partial DE, \partial PD$.

State 11 is the only potentially stable state of the model. As one can see in Table 14, one or two main transition causes of rank 1 are pending at every other state. No tardy tendency switch is pending at state 11. Therefore state 11 is ex ante stationary. The negative, and positive perturbances of ∂IN are the expected perturbances at state 11.

Tables 30 and 31 examine the consequences of the expected perturbances. Since the model of Table 6 is fully anchored only activity 1 is used in the realizations shown in these tables.

Therefore the column "activity" is omitted in Tables 30 and 31 as well as in the case of four other tables which will be discussed later. There is a uniquely determined reentry history after each of the two perturbances. The reentry state is state 8 in the case of $[\partial IN : +]$ and state 14 for $[\partial IN : -]$. At these two states a lag extinction of ∂PD^{-} has rank 1 (see Table 14). These lag extinctions do not lead back to state 11 but to the states 9 and 13, respectively (see Figure 4 and 4.10.2). Therefore the stationary state 11 is not stable.

However, the fact that the reentry state after an expected perturbance is in the cycle does not permit us to conclude that the stationary state is a repulsor. We cannot yet exclude the possibility that the transition diagram has rank 2 and that a permissible path of rank 2 leaves the cycle and eventually returns to state 11. Therefore it will now be shown that the rank of the transition diagram is 1.

At each of the states 1, 3, 5, 8, 9, 12, 17, 21, 19, 14, 13, 10, and 5 of the cycle (see Figure 4) exactly one main transition cause of rank 1 is pending. The transition due to this cause leads to the next state of the cycle. At states 8 and 14 a tardy shift of PD to n is pending, which, however, does not stay unresolved, since it becomes effective one step later (see Figure 4). Therefore a permissible path of rank 1 begins at every state of the cycle.

Apart from the stationary state 11 there are seven states outside the cycle, namely the states 2, 4, 6, 7, 15, 16, and 18. Of course, at state 11 a permissible path of rank 1 begins which also ends at this state. It remains to show that at each of the other 8 states outside the cycle a permissible path of rank 1 begins.

Tables 32 - 35 show all tentative paths of rank 1 beginning with one of the eight states up to a state of the cycle.

Each of these paths leads to a state of the cycle after at most two steps. The Tables 32 - 35 also show realizations of the readjustment process for transitions along these paths. After a path of rank 1 has reached the cycle it runs through the cycle again and again. Since no shift or lag extinction remains unresolved along the cycle it is clear that all these paths are permissible. We can conclude that the rank of the transition diagram is 1.

With the Tables 32 - 35 we have gained a complete overview over all paths of rank 1. This is sufficient for drawing the transition diagram. Figure 11 shows the extended transition diagram. This diagram permits the conclusion that the stationary state is a repulsor.

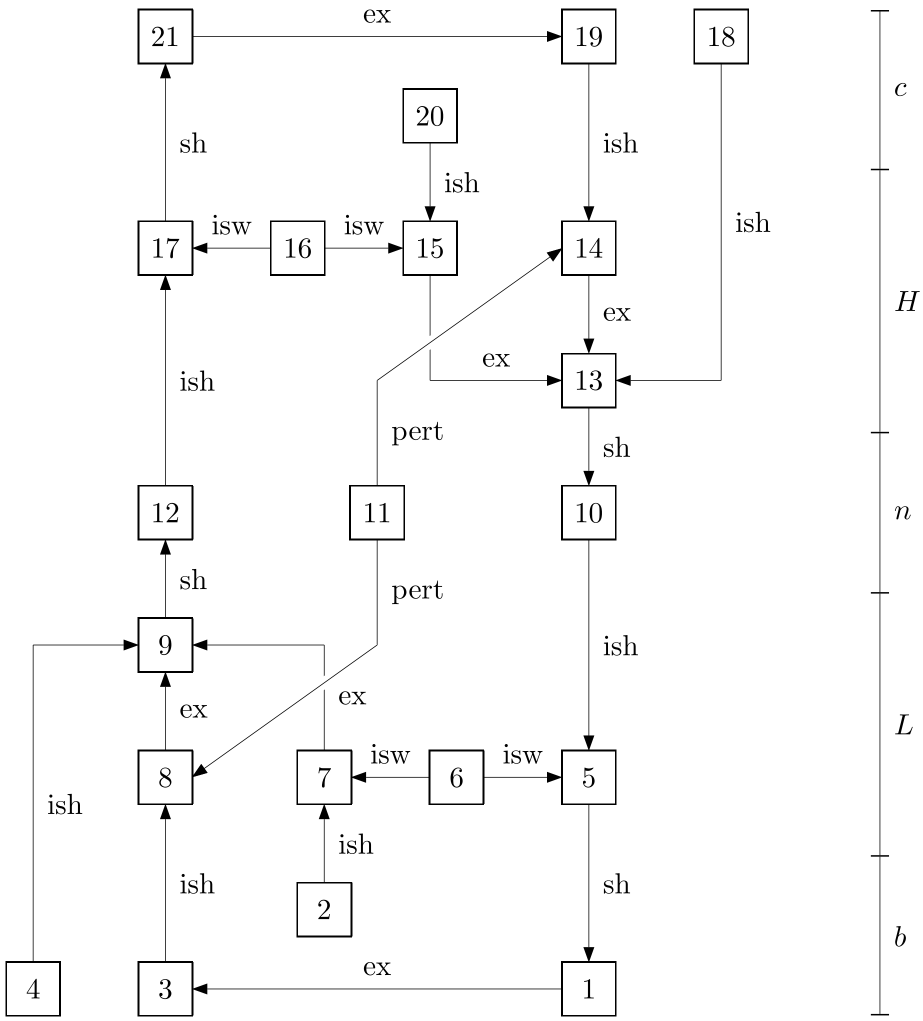


Figure 11. The extended transition diagram for the model of Table 6

Abbreviations

ex	lag extinction	isw	immediate switch
ish	immediate shift	sh	tardy shift
pert	perturbance		

base	comment	TR	∂GO	∂DE	∂PR	∂IM	∂EX	∂TR
original	state 2	b	0	0	0	0	0	0
auxiliary	$[\partial GO : +]$	b	00 ++F	00 ++F	00 ++F	00 ++F	00 $--$F	00 $--$F
	state a_0	b	+	+	+	+	$-$	$-$
	$[TR \to D]$ immediate shift	D	++ ++F	++ ++F	++ ++F	++ ++F	$--$ $--$F	$--$ $--$F
	state a_1	D	+	+	+	+	$-$	$-$
original	return start	D	++ $--$F	++ $--$F	++ $--$F	++ $--$F	$--$ ++F	$--$ ++F
	reentry state 1	D	$-$	$-$	$-$	$-$	+	+
	$[TR \to b]$ tardy shift	b	$--$ 00F	$--$ 00F	$--$ 00F	$--$ 00F	++ 00F	++ 00F
	state 2	b	0	0	0	0	0	0

TABLE 27. Return to the stationary state after $[\partial GO : +]$

base	comment	PD	$\square DE$	∂IN	∂DE	∂PD	activity
original	stationary state 9	n	$\{-,0,+\}$	0	0	0	
auxiliary	perturbance	n	$\{-,0,+\}$	00	00	00	
	$[\partial IN : +]$		$\{-,0,+\}$F				1
				$++$F			1
					-0		3
					$--$F		4
						$--$F	1
	state a_0	n	$\{-,0,+\}$	$+$	$-$	$-$	
	$[PD \to L]$	L	$\{-,0,+\}$	$++$	$--$	$--$	
	immediate shift		$\{-,0,+\}$F				1
				$++$F			1
					$--$F		4
						$--$F	4
	state a_1	L	$\{-,0,+\}$	$+$	$-$	$-$	
original	return start	L	$\{-,0,+\}$	$++$	$--$	$--$	
			$\{-,0,+\}$F				1
				$--$F			1
					$--$F		4
						$--$F	4
	reentry state 8	L	$\{-,0,+\}$	$-$	$-$	$-$	

TABLE 28. Destabilization by $[\partial IN : +]$ in the model of Table 4

base	comment	PD	$\square DE$	∂IN	∂DE	∂PD	activity
original	stationary state 9	n	$\{-,0,+\}$	0	0	0	
auxiliary	perturbance	n	$\{-,0,+\}$	00	00	00	
	$[\partial IN:-]$		$\{-,0,+\}$F				1
				$--$F			1
					$+0$		3
					$++$F		4
						$++$F	1
	state a_0	n	$\{-,0,+\}$	$-$	$+$	$+$	
	$[PD \to H]$	H	$\{-,0,+\}$	$--$	$++$	$++$	
	immediate shift		$\{-,0,+\}$F				1
				$--$F			1
					$++$F		4
						$++$F	4
	state a_1	H	$\{-,0,+\}$	$-$	$+$	$+$	
original	return start	H	$\{-,0,+\}$	$--$	$++$	$++$	
			$\{-,0,+\}$F				1
				$++$F			1
					$++$F		4
						$++$F	4
	reentry state 4	H	$\{-,0,+\}$	$+$	$+$	$+$	

TABLE 29. Destabilization by $[\partial IN:-]$ in the model of Table 4

base	comment	PD	∂PD^{-}	$\square DE$	∂IN	∂DE	∂PD
original	state 11	n	0	$\{-,0,+\}$	0	0	0
	perturbance $[\partial IN : +]$	n	0	$\{-,0,+\}$ $\{-,0,+\}$F	00 ++F	00 −−F	00 −−F
auxiliary	opening state a_0	n	0	$\{-,0,+\}$	$+$	$-$	$-$
	$[PD \to L]$ immediate shift	L	0	$\{-,0,+\}$ $\{-,0,+\}$F	++ ++F	−− −−F	−− −−F
	state a_1	L	0	$\{-,0,+\}$	$+$	$-$	$-$
original	return start	L	0	$\{-,0,+\}$ $\{-,0,+\}$F	++ −−F	−− ++F	−− ++F
	reentry state 8	L	0	$\{-,0,+\}$	$-$	$+$	$+$

TABLE 30. Destabilization by $[\partial IN : +]$ in the model of Table 6

base	comment	PD	∂PD^{-}	$\Box DE$	∂IN	∂DE	∂PD
original	state 11	n	0	$\{-,0,+\}$	0	0	0
	perturbance $[\partial IN : -]$	n	0	$\{-,0,+\}$ $\{-,0,+\}$F	00 $--$F	00 $++$F	00 $++$F
auxiliary	opening state a_0	n	0	$\{-,0,+\}$	$-$	$+$	$+$
	$[PD \to H]$ immediate shift	H	0	$\{-,0,+\}$ $\{-,0,+\}$F	$--$ $++$F	$++$ $--$F	$++$ $--$F
	state a_1	H	0	$\{-,0,+\}$	$+$	$-$	$-$
original	return start	H	0	$\{-,0,+\}$ $\{-,0,+\}$F	$++$ $++$F	$--$ $--$F	$--$ $--$F
	reentry state 14	H	0	$\{-,0,+\}$	$+$	$-$	$-$

TABLE 31. Destabilization by $[\partial IN : -]$ in the model of Table 6

base	comment	PD	∂PD^-	$\Box DE$	∂IN	∂DE	∂PD
	state 2	b	$-$	$\{0,+\}$	$-$	$+$	$+$
	$[PD \to L]$	L	$-$	$\{0,+\}$	$--$	$++$	$++$
	immediate			$\{-,0,+\}$F			
	shift				$--$F		
						$++$F	
							$++$F
original	state 7	L	$-$	$\{-,0,+\}$	$-$	$+$	$+$
	$[\partial PD^-]$	L	$+$	$\{-,0,+\}$	$--$	$++$	$++$
	lag extinction			$\{-,0,+\}$F			
					$--$F		
						$++$F	
							$++$F
	state 9 (cycle)	L	$+$	$\{-,0,+\}$	$-$	$+$	$+$

	state 4	b	$+$	$\{0,+\}$	$-$	$+$	$+$
	$[PD \to L]$	L	$+$	$\{0,+\}$	$--$	$++$	$++$
original	immediate			$\{0,+\}$F			
	shift				$--$F		
						$++$F	
							$++$F
	state 9 (cycle)	L	$+$	$\{0,+\}$	$-$	$+$	$+$

TABLE 32. Paths of rank 1 into the cycle from states 2, 4, and 7

base	comment	PD	∂PD^{-}	$\Box DE$	∂IN	∂DE	∂PD
	state 20	c	$+$	$\{-,0\}$	$+$	$-$	$-$
	$[PD \to H]$	H	$+$	$\{-,0\}$	$++$	$--$	$--$
	immediate			$\{-,0,+\}$F			
	shift				$++$F		
						$--$F	
							$--$F
original	state 15	H	$+$	$\{-,0,+\}$	$+$	$-$	$-$
	$[\partial PD^{-}]$	H	$-$	$\{-,0,+\}$	$++$	$--$	$--$
	lag extinction			$\{-,0,+\}$F			
					$++$F		
						$--$F	
							$--$F
	state 13 (cycle)	H	$-$	$\{-,0,+\}$	$+$	$-$	$-$

	state 18	c	$-$	$\{-,0\}$	$+$	$-$	$-$
	$[PD \to H]$	H	$-$	$\{-,0\}$	$++$	$--$	$--$
original	immediate			$\{-,0,+\}$F			
	shift				$++$F		
						$--$F	
							$--$F
	state 13 (cycle)	H	$-$	$\{-,0,+\}$	$+$	$-$	$-$

TABLE 33. Paths of rank 1 into the cycle from states 20, 15, and 18

base	comment	PD	∂PD^-	$\square DE$	∂IN	∂DE	∂PD
original	state 6	L	$-$	$\{-,0,+\}$	$-$	0	0
hypothetical	$[\partial DE \to -]$	L	$-$	$\{-,0,+\}$	$--$	00	00
	immediate			$\{-,0,+\}$F			
	switch				$--$F		
						$--$F	
							$--$F
original	state 5 (cycle)	L	$-$	$\{-,0,+\}$	$-$	$-$	$-$

original	state 6	L	$-$	$\{-,0,+\}$	$-$	0	0
hypothetical	$[\partial DE \to +]$	L	$-$	$\{-,0,+\}$	$--$	00	00
	immediate			$\{-,0,+\}$F			
	switch				$--$F		
						$++$F	
							$++$F
original	state 7 *	L	$-$	$\{-,0,+\}$	$-$	$+$	$+$

TABLE 34. Paths of rank 1 into the cycle from state 6

*State 7 leads to state 9 (see Table 32)

base	comment	PD	∂PD^{-}	$\Box DE$	∂IN	∂DE	∂PD
original	state 16	H	$+$	$\{-,0,+\}$	$+$	0	0
hypo- theti- cal	$[\partial DE \to -]$ immediate switch	H	$+$	$\{-,0,+\}$ $\{-,0,+\}$F	$++$ $++$F	00 $--$F	00 $--$F
original	state 15 *	H	$+$	$\{-,0,+\}$	$+$	$-$	$-$

original	state 16	H	$+$	$\{-,0,+\}$	$+$	0	0
hypo- theti- cal	$[\partial DE \to +]$ immediate switch	H	$+$	$\{-,0,+\}$ $\{-,0,+\}$F	$++$ $++$F	00 $++$F	00 $++$F
original	state 17 (cycle)	H	$+$	$\{-,0,+\}$	$+$	$+$	$+$

TABLE 35. Paths of rank 1 into the cycle from state 16

*State 15 leads to state 13 (see Table 35)

CHAPTER 6

Reduction

6.1. The problem

It may happen that a qualitative dynamic system contains two variables, one of which is just a duplication of the other. As an example consider the variable DE in the model for Hume's specie flow mechanism. We always have

$$\partial DE = \partial GO$$

This suggests that the variable ∂DE is superfluous and can be eliminated. Elimination of ∂DE means that the confluence for ∂DE is removed from the system and that ∂DE is replaced by ∂GO wherever ∂DE appears on the right hand side of another confluence or a restriction equation. In the case of the model for Hume's specie flow mechanism the confluence

$$\partial PR = \partial DE$$

is thereby changed to

$$\partial PR = \partial GO$$

Since ∂DE does not appear on the right hand side of another confluence, the confluences for $\partial IM, \partial EX, \partial TR$ and ∂GO remain unchanged. Elimination of ∂DE leads to a "reduced system". It is reasonable to expect that the analysis of the reduced system leads to essentially the same result as that of the original one. The tendency ∂DE is not more than an intermediate link between ∂GO and ∂PR. It should not matter whether DE is explicitly modelled or not. The results of the analysis should not depend on unimportant modelling details. This is actually the case for the elimination of DE in the model for Hume's specie flow mechanism. However, it is by no means obvious, under which conditions a variable can be eliminated without changing the results of the analysis.

In the next section the notion of a "removable" variable will be introduced. It will be argued that the elimination of a removable variable amounts to a change of unimportant modelling detail and that for other variables the same is not necessarily true. It seems to be an indispensible requirement for a theory of qualitative reasoning about economic dynamics that removable variables can be eliminated without any adverse effects on the conclusions reached by the analysis. It is the aim of this chapter to show that the theory proposed here meets this requirement.

The elimination of removable variables can considerably simplify the analysis. Successive eliminations of DE, PR, EX and IM in this order transform the model for Hume's specie flow mechanism to a system with just two variables, TR and GO.

6.2. Removability and eliminability

6.2.1. Definitions. All definitions of this chapter refer to a fixed but arbitrary base $B = (\Lambda, \Gamma)$ or to a fixed but arbitrary qualitative dynamic system

$$\Phi = (\Lambda, \Gamma, \rho, \alpha)$$

with this base. A confluence is called **scale independent**, if its right hand side does not depend on values of scaled variables. A variable is called **restricted**, if the main term of its confluence is subject to a boundary restriction or a system specific restriction. Otherwise it is **unrestricted**. A variable is called **restrictive** if its tendency, its boundary restriction, or its system specific restriction appears on the right hand side of a restriction equation. Otherwise it is **unrestrictive**.

A variable is **lag free**, if its lagged tendency does not appear on the right hand side of any confluence or restriction equation. A variable is **perturbance free**, if for no potentially stationary state s a perturbance of the tendency of this variable is in $\alpha(s)$. A confluence is **monocausal**, if its main term has only one component, which may be a constant direction, a signed lagged tendency or a signed current tendency.

A variable is called **alone** if it is the only one in the system. We say that the variable RV is **short looped** if a tendency ∂XY appears in the main term of the confluence for ∂RV and ∂RV appears in the main term of the confluence for ∂XY or the restriction equation for $\Box XY$. (The letters R and V are the initials of the words "removable" and "variable".)

A variable RV is called **removable** in Φ if it satisfies the following eight **removability conditions:**

(e1) The confluence for ∂RV is scale independent
(e2) The variable RV is unscaled
(e3) The variable RV is unrestricted
(e4) The variable RV is lag free
(e5) The confluence for ∂RV is monocausal
(e6) The variable RV is not alone
(e7) The variable RV is not short looped
(e8) The variable RV is perturbance free

A variable is called **eliminable** in $B = (\Lambda, \Gamma)$ if it satisfies all removability conditions with the exception of (e8). Whether the removability conditions (e1) to (e7) are satisfied or not depends only on the base $B = (\Lambda, \Gamma)$ of Φ. However, a perturbance free variable is defined by the property that for no potentially stationary state s the tendency of this variable is in $\alpha(s)$. Therefore it does not only depend on the base B but also on the perturbance assignment α whether (e8) is satisfied or not.

In this chapter the main interest is the elimination of removable variables, but many intermediary results stated as lemmas are more naturally formulated and more easily derived as statements about eliminable variables. Since every removable variable is also eliminable one gains valuable insights into removability by the investigation of eliminability.

It will be important to distinguish between two kinds of eliminable variables. An eliminable variable EV is a **source** if the confluence for ∂EV has one of the following three forms:

(f1) $\partial EV = \partial XY^-$
(f2) $\partial EV = -\partial XY^-$
(f3) $\partial EV = d$

where d is a constant direction. In view of (e4) the variable XY cannot be the variable EV. An eliminable variable EV is a **link** if it has one of the following two forms:

(f4) $\partial EV = \partial XY$
(f5) $\partial EV = -\partial XY$

In both cases the variable XY whose tendency appears on the right hand side is called the **determinator** of EV. In view of the removability conditions (e1), (e3), (e4), and (e5) it is clear that an eliminable variable is either a source or a link.

Remark. *A removable or eliminable variable EV need not be unrestrictive. It is permitted that ∂EV appears in a restriction equation for a system specific restriction $\Box WZ$, where WZ is not the variable EV. Of course, in this case WZ cannot be the determinator of EV since otherwise EV would be short looped.*

6.2.2. Interpretation of the removability conditions. An acceptable theory of qualitative reasoning about economic dynamics should have the property that the results of the analysis do not depend on arbitrary modelling details. Whether a removable variable is explicitly modelled or not is often an arbitrary modelling decision. Thus it is not essential for the model of Hume's specie flow mechanism whether the variable DE is explicitly modelled or not.

It will now be argued that variables which are not removable cannot be eliminated or should not be eliminated. A scale dependent confluence embodies modelling details connected to distinctions between combinations of values of scaled variables. Such distinctions are usually essential features which cannot be removed without changing the character of the model. Therefore condition (e1) is required. Obviously (e2), (e3), and (e4) have similar justifications.

As will be explained in more detail in the next section, elimination of an eliminable variable EV means that first on the right hand side of every confluence or restriction equation ∂EV is replaced by the right hand side of the confluence for ∂EV. Then some equivalent transformations are applied to the main terms of confluences and restriction equations. Thereby the main terms receive a form which satisfies the requirements (c1) to (c10) on the structure of confluences and restriction equations (see 2.8).

If the confluence for ∂EV is monocausal then the replacement of ∂EV by the right hand side of its confluence is a substitution of one direction by another equal direction. This is not the case if the main term of the confluence for ∂EV is a sum of several components. The equality sign in a confluence means that the left hand side is an element of the right hand side. In an algebraic equation left hand side and right hand side are always equal, but this is not the case for a confluence unless it is monocausal. Therefore it is necessary to require (e5).

If EV is alone, it cannot be eliminated, since by definition the list of variables must be non-empty. Therefore (e6) is required. Suppose that EV satisfies (e1) to (e6) but not (e7). Assume, for example that the confluence for ∂EV is $\partial EV = \partial XY$. If EV is short looped then ∂EV appears in the main term of the confluence for ∂XY or in the restriction equation for $\Box XY$. Elimination of EV would result in a confluence for ∂XY violating (c9) of 2.8 in the first case or in a restriction equation for $\Box XY$ violating (c10) in the second case. Therefore (e7) must be required.

If a perturbance of an eliminable variable ∂EV is expected at a stationary state s, then this is not an unimportant modelling detail. Therefore EV is not considered to be removable unless it satisfies (e8).

6.3. Elimination

In this section it will be explained how the list of variables and the list of confluences and restriction equations are changed by the elimination of an eliminable variable. This is the first step towards the definition of a "reduced base after the elimination of EV".

Let EV be an eliminable variable. The **reduced list of variables after the elimination of** EV is defined as follows: The list Λ' contains all variables in the list Λ with the exception of RV and no other ones. The scaled variables have the same scales in Λ and Λ'. The list Λ' is also referred to as the **reduction of** Λ **after the elimination of** RV.

The elimination of EV concerns only the main terms of confluences and restriction equations. The restriction to which a main term is accomodated remains the same. It will now be explained how the main terms are changed.

Let R be the right hand side of the confluence for ∂EV and let WZ be a variable which is different from EV. Let T be the main term of the confluence for ∂WZ or the main term of the restriction equation for $\Box WZ$. The **result T_0 of substituting R for ∂EV in T** is the expression obtained from T by replacing ∂EV by R and $-\partial RV$ by $-R$ while all other components of T remain unchanged.

The result of substituting R for EV generally does not satisfy (c1) to (c10) of 2.8 and therefore cannot serve as the main term after the elimination of EV. Simplifying equivalent transformations have to be applied. Two of these transformations, the summation of constants and the deletion of variable components have been already introduced in 3.3.1 in connection with the definition of the auxiliary base for a perturbance. The elimination of an eliminable variable involves two further simplifying equivalent transformations.

A constant component of an algebraic sum S will simply be called a **constant in S**. Similary a **zero in S** is a constant component of S with the value zero. The following four transformations, including the two introduced already in 3.3.1 are aplied to expressions T_i appearing in a sequence leading from T_0 to the new main term T'.

1. **Summation of constants:** If T_i has several constant components then all of them are replaced by the convex direction set which is their sum
2. **Deletion of zero:** If T_i has at least one variable component and exactly one constant component whose value is zero then this constant component is deleted
3. **Deletion of variable components:** If T_i has at least one variable component and exactly one constant component with the value $\{-, 0, +\}$ then all variable components are deleted
4. **Deletion of duplicates:** If T_i has doubly represented variable components, then one component in every pair of equal components is deleted.

Each of the four transformations has a **condition of applicability** spelled out by the if-phrase before the description of how T_i is changed. We say that a transformation is **applicable to T_i**, if its applicability condition is satisfied. The transformations are applied, one after the other in the order in which they are listed above, each of them at most once, as far as they are applicable. The result of this procedure is the **reduction T' of T after the elimination of EV**.

The **reduced confluence for ∂WZ** or the **reduced restriction equation for $\Box WZ$ after the elimination** of RV is obtained by replacing the main term T of the concerning confluence or restriction equation by its reduction T' leaving everything else unchanged. The **reduced list Γ' of confluences and restriction equations** or the **reduction of Γ after the elimination of** RV contains all reductions of confluences of tendencies of variables in Λ' after the elimination of EV and all reduced restriction equations after the elimination of EV derived from the restriction equations in Γ. The pair (Λ', Γ') is called the **reduced base** or the **reduction** of (Λ, Γ) **after the elimination of** EV.

It has to be shown that the reduction T' of a main term in Γ is an expression T_i to which none of the four transformations is applicable. This will be the content of lemma 23.

Table 36 shows how many constants are in T_0. If R is not constant then T_0 has exactly as many constants as T. If R is constant, then the substitution of ∂EV and $-\partial EV$ by R or $-R$, respectively, may bring in one or two new constants depending on whether ∂EV or $-\partial EV$ are components of T. Obviously Table 36 correctly indicates the number of constants in T_0.

	R is constant		R is not constant	
Among the components of T are	no constant in T	one constant in T	no constant in T	one constant in T
Neither ∂EV nor $-\partial EV$	[1] 0	[2] 1	[3] 0	[4] 1
Either ∂EV or $-\partial EV$	[5] 1	[6] 2	[7] 0	[8] 1
∂EV and $-\partial EV$	[9] 2	[10] 3	[11] 0	[12] 1

TABLE 36. Number of constants in T_0

If the right hand side R of the confluence for ∂EV is a current or lagged tendency then the substitution of ∂EV and $-\partial EV$ by R or $-R$, respectively, may bring in pairs of doubly represented components, if R or $-R$ are components. It is clear that no component of T_0 can be represented more than twice. Table 37 shows the number of pairs of doubly represented components in T_0. Obviously this number is correctly indicated.

	R is a current or lagged tendency				R is a constant
	Neither R nor $-R$ in T	R but not $-R$ in T	$-R$ but not R in T	R and $-R$ in T	
Neither ∂EV nor $-\partial EV$ in T	[1] 0	[2] 0	[3] 0	[4] 0	[5] 0
∂EV but not $-\partial EV$ in T	[6] 0	[7] 1	[8] 0	[9] 1	[10] 0
$-\partial EV$ but not ∂EV in T	[11] 0	[12] 0	[13] 1	[14] 1	[15] 0
∂EV and $-\partial EV$ in T	[16] 0	[17] 1	[18] 1	[19] 2	[20] 0

TABLE 37. Number of pairs of doubly represented components of T_0

Table 38 employs a case distinction according to the numbers of constant and variable components, the presence and absence of duplicates, and the value of the sum of all constants in T_0. For each of 10 cases it is shown which of the four transformations have to be applied one after the other until the reduction T' of T is reached. It will be discussed in the proof of Lemma 23 why the entries of Table 38 are correct.

For the sake of simplicity we shall sometimes refer to the four transformations by the number in their order of application. Thus summation of constants is transformation 1, deletion of zero is transformation 2, deletion of variable components is transformation 3 and deletion of duplicates is transformation 4.

Tables 36, 37 and 38 contain case numbers in the upper right corner of fields. This will provide an easy way of referring to individual cases.

The four transformations, summation of constants, deletion of zero, deletion of variable components, and deletion of variables are **equivalent transformations** in the sense that for any fixed specifications of values for the pieces in a main term to which they are applicable they do not change the value of the main term.

<table>
<tr><th rowspan="2">constants in T_0</th><th colspan="4">variable components of T_0</th></tr>
<tr><th>none</th><th colspan="3">at least one</th></tr>
<tr><td rowspan="2">none</td><td rowspan="2">not possible</td><td colspan="2">no duplicates</td><td>duplicates</td></tr>
<tr><td colspan="2">[1]</td><td>[2] deletion of duplicates</td></tr>
<tr><td rowspan="3">one</td><td rowspan="3">[3]</td><td colspan="2">value of constant</td><td rowspan="3">[6] deletion of duplicates</td></tr>
<tr><td>$\{0\}$</td><td>not $\{0\}$</td></tr>
<tr><td>[4] deletion of zero</td><td>[5]</td></tr>
<tr><td rowspan="4">more than one</td><td colspan="4">summation of constants</td></tr>
<tr><td rowspan="3">[7]</td><td colspan="3">sum of constants</td></tr>
<tr><td>$\{0\}$</td><td>$\{-,0,+\}$</td><td>others</td></tr>
<tr><td>[8] deletion of zero</td><td>[9] deletion of variable components</td><td>[10]</td></tr>
</table>

TABLE 38. Transformations applied to T_0

LEMMA 23. *Let T be the main term of a confluence or restriction equation of Γ and let T_0 be the result of substituting the tendency ∂EV of an eliminable variable EV in T. Moreover let T' be the reduction of T after the elimination of EV. Then none of the four transformations is applicable to T'. Table 38 shows by which successive transformations T' results from T.*

PROOF. If there is no constant in T_0 then the transformations 1, 2 and 3 are not applicable to T_0 since they presuppose the presence of at least one constant. Transformation 4 cannot be applied unless there are duplicates in T_0. Therefore, in case 1 of Table 38 the reduction T' is nothing else than T_0. In case 2 only the deletion of duplicates is possible and the result T_1 of applying transformation 4 to

T_0 is the reduction T'. In both cases it is clear that none of the four transformations is applicable to T'.

Assume that there is exactly one constant in T_0. In case 3 of Table 38 there is no other component in T_0. Therefore in this case T' is nothing else than T_0 and none of the four transformations can be applied to T'. Suppose that there is at least one variable component in T_0 but no doubly represented variable components. Table 36 shows that this situation can arise if there is a constant in T and in case 5 of Table 36. If there is a constant in T it is also the constant in T_0. In case 5 of Table 36 the constant arises by the substitution of either ∂EV or $-\partial EV$ by a constant R. Since T_0 has at least one variable component, T has at least one variable component, too. Therefore it follows by (c6) and (c7) that the constant of T, if it has one, is unequal to $\{0\}$ and to $\{-, 0, +\}$. However in case 5 of Table 36 the constant in T_0 is $\{0\}$ for $\partial EV = 0$. Therefore Table 38 distinguishes between the cases 4 and 5. In case 4 of Table 38 the transformation deletion of zero is applied to T_0. This yields an expression T_1 to which none of the four transformations is applicable, since T_1 has no constant and no duplicates. Therefore T' is nothing else than T_1 in case 4 of Table 38. In case 5 of Table 38 the constant of T_0 is unequal to $\{-, 0, +\}$, since it is either the constant of T or R or $-R$. Therefore in this case none of the four transformations can be applied to T_0 and T' is T_0.

Now consider case 6 of Table 38. Duplicates cannot arise unless R is variable. Therefore the constant of T_0 must be the constant of T. As we have seen above this constant is different from $\{0\}$ and from $\{-, 0, +\}$. Therefore none of the four transformations can be applied to T_0 and T' is T_0 in this case.

Assume that there are at least two constants in T_0. In all cases of this kind summation of constants is applied to T_0. Let T_1 be the expression obtained by this. T_1 has no variable components if T_0 has no variable components. Therefore in case 7 of Table 38 none of the four transformations can be applied to T_1 and T' is T_1.

Now consider the cases 8, 9, and 10 of Table 38. There cannot be any duplicates if there are more than one constant in T_0 since only one constant can come from T but the others must arise as a consequence of the substitution of ∂EV by a constant R. Therefore deletion of duplicates cannot be applied in the cases 8, 9, and 10 of Table 38. In case 8 of Table 38 deletion of zero is applied and in case 10 deletion of variable components. It is clear that in each of the two cases an expression T_1 is obtained to which none of the four transformations is applicable. T' is this expression T_1. This completes the proof of the lemma. □

Comment. *It has been shown how the elimination of EV transforms a main term T to a new main term T'. However, this does not yet answer the question* *whether the reduced base (Λ', Γ') defined above is a base in the sense of the definition in 2.9. A first step in this direction was lemma 23. A further step will be lemma 24. Finally lemma 25 will give a positive answer to the question.*

LEMMA 24. *Under the assumptions of lemma 23 the following assertions (1), (2) and (3) hold*

(1) *T' satisfies (c3), (c6) and (c7).*
(2) *If T is the main term of a confluence in Γ then T' has the properties required by (c4).*
(3) *If T is the main term of a restriction equation in Γ then T' has the properties required by (c5) and (c8).*

PROOF. of (1): T satisfies (c3) and therefore has at least one component and finitely many variable components. Substitution of ∂EV and $-\partial EV$ by R and $-R$, resp., does not change the number of components but never deletes all of them. Therefore also T' has at least one component and finitely many variable components. As we have seen before no variable component is represented more than twice in T_0. Variable components can be deleted by the transformations 2, 3, and 4 but no new ones can arise. Therefore no variable component can be represented more than twice in T' either. In view of lemma 23 transformation 4 is not applicable to T'. Therefore no component of T' is represented more than twice in T'. If there are more than one constants in T_0 the number of constants is reduced to one by transformation 1. We have shown that T' satisfies (c3).

In view of lemma 23 transformation 2 is not applicable to T'. Therefore T' satisfies (c6). Similarly transformation 3 is not applicable to T'. Therefore T' satisfies (c7). □

PROOF. of (2): Let T be the main term of a confluence in Γ. Then T satisfies (c4) and (c9). Therefore the constant in T, if there is one is a direction sum. If R is constant then T_0 may have up to three constant components. The constant components due to the substitution of ∂EV by R are directions. Therefore the sum of all constants in T_0 is a direction sum. Therefore a constant in T' is a direction sum. Variable components of T' are either variable components of T or they are due to the substitution of ∂EV by R. If R is variable then it is a current or lagged tendency. Since T satisfies (c4) all variable components of T' are current or lagged tendencies. Consequently T' satisfies (c4). □

PROOF. of (3): Let T be the main term of a restriction equation in Γ. Then T satisfies (c5). Since a sum of convex direction sets is a convex direction set the constant in T' must be convex direction sets. The variable components of T' are either in T or they are due to the substitution of ∂EV by a current or lagged tendency R. Therefore (c5) is satisfied for T'.

Boundary restrictions are neither affected by the substitution of ∂EV by R nor by the four transformations. Since T satisfies (c8) it is clear that $\triangleright EV$ and $- \triangleright EV$ are not both in T. The same must be true for T'. Therefore (c8) is satisfied for T'. □

LEMMA 25. *The reduced base* (Λ', Γ') *of* (Λ, Γ) *after the elimination of an eliminable variable* EV *is a system base as defined in 2.9.*

PROOF. Elimination of EV removes EV from the list of variables and the confluence for ∂EV from the list of confluences and restriction equations. All other confluences and restriction equations are changed to reduced ones. It is clear that Λ' has the properties of a list of variables and that Γ' satisfies (b1), (b2) and (b3) of 2.7.

Let T be a main term of a confluence or restriction equation in Γ other than the confluence for ∂EV and let T' be the reduction of T after the elimination of EV. The reduced confluences and restriction equations in Γ' differ from their counterparts in Γ only with respect to their main terms but not with respect to their restrictions. Since (c1) and (c2) are satisfied for Γ it follows that (c1) and (c2) are also satisfied for Γ'.

Lemma 24 shows that (c3) to (c8) are satisfied for confluences and restriction equations in Γ'. It remains to show that the same is true for (c9) and (c10).

(c9) and (c10) only concern the case that EV is a link and that ∂EV or $-\partial EV$ are components of the main term of the confluence for a tendency ∂XY or of the restriction equation for $\Box XY$ in Γ. Property (c7) of an eliminable variable excludes the possibility that this situation can arise. Therefore conditions (c9) and (c10) are satisfied.

It remains to show that (Λ', Γ') satisfies the anchoring requirement (see 2.9). Suppose that ∂EV is not anchored in (Λ, Γ). Then each directional which is anchored in Γ has the same confluence or restriction equation in Γ and Γ'. It follows that the anchoring requirement is satisfied for (Λ', Γ').

Now assume that ∂EV is anchored. Then the right hand side R of the confluence for ∂EV is also anchored in Γ. Moreover R has a lower anchorage level than ∂EV. A directional anchored in Γ and different from ∂EV remains anchored in Γ' if on the right hand side of its confluence or restriction equation ∂EV is replaced by R. This is not changed by the later application of transformations 1 to 4. It follows that the anchoring requirement is satisfied for (Λ', Γ'). This completes the proof of the lemma. □

6.4. The state mapping

As before let EV be an eliminable variable of $B = (\Lambda, \Gamma)$ and let $B' = (\Lambda', \Gamma')$ be the reduction of B after the elimination of EV in B. In this section a one-to-one mapping from the states of B onto the states of B' will be introduced. This "state mapping" will be very important for the remainder of this chapter. It will enable us to define a priority ranking and auxiliary priority function for the reduced base and it will be essential for the derivation of results.

Consider a state s for B. If the specification of ∂EV is taken out of s one obtains a specification of the values of all scaled variables in Λ', all current tendencies of variables in Λ', and all lagged tendencies and system specific restrictions appearing in confluences and restriction equations of Γ'. This is so since ∂EV^- and $\Box EV$ do not appear in Γ. We call s' the **reduction of s after the elimination of EV**. In this way a reduction s'

$$s' = \lambda(s)$$

is used in order to express the relationship between a state s and its reduction s' after the elimination of EV. Since EV is kept fixed the dependence of s' on EV is not made explicit for the sake of simplicity. For the same reason, we shall often drop the phrase "after the elimination of EV" and simply speak about reductions of main terms, confluences and restriction equations in contexts, in which EV is kept fixed.

It is clear that s' has the properties (a1) to (a3) required for a state in 2.7. As we shall see s' also has the property (a4) with respect to Γ'. Therefore s' is a state for (Λ', Γ'). For this reason λ is called the **state mapping for the elimination of EV in B** or simply the state mapping in contexts in which B and EV are kept fixed.

In 3.1 transition causes have been introduced as formal objects by expressions in rectangular brackets. The same expression may describe transition causes at different states for the same base or even for different bases. In this sense one speaks of the same transition cause pending at a state s and its image $\lambda(s)$.

A shift of EV cannot be pending at a state s since EV is unscaled by (e2). A lag extinction of ∂EV^- is impossible in view of (e4). A tendency switch of ∂EV cannot be pending at a state s since the confluence for ∂EV is monocausal. We can conclude that no main transition causes of ∂EV can be pending at any state s.

Perturbances of ∂EV can be pending at a stationary state s. Even if EV is not only eliminable but removable in a qualitative dynamic system Φ with the base B this may happen. The removability condition (e8) only requires that no expected perturbances of a tendency of a removable variable are pending at a stationary state. Of course, perturbances of EV cannot be pending at any state of the reduction (Λ', Γ') after the elimination of EV.

We say that a transition cause or halfway switch ω is **invariant under the state mapping** λ if the following is true: ω is pending at a state $s' = \lambda(s)$ if and only if ω is pending at s. It will be the aim of this section to show that all transition causes with the exception of perturbances of ∂EV and all halfway switches are invariant under the state mapping. This will be the content of lemma 27.

LEMMA 26. *Let EV be an eliminable variable of a base $B = (\Lambda, \Gamma)$ and let $B' = (\Lambda', \Gamma')$ be the reduction of B after the elimination of EV. Moreover let s be a state for B and let*

$$s' = \lambda(s)$$

be the reduction of s assigned to s by the state mapping λ for the elimination of EV in B. Then s' is a state of B' and λ is a one-to-one mapping from the set of all states of B onto the set of all states of B'.

PROOF. We first show that s' is a state for B'. Let T be the main term of a confluence or restriction equation other than the confluence for ∂EV in Γ. As before let R be the right hand side of the confluence for ∂EV and let T_0 be the result of substituting ∂EV by R in T.

It can be seen immediately, that T_0 has the same value as T at s. Since the four transformations 1 to 4 are equivalent transformations, it is clear that T', too, has the same value as T at s. The boundary restrictions have the same values in B and in B'. It follows that the right hand side of a reduced confluence or restriction equation has the same value as the right hand side of the original confluence or restriction equation. Therefore the definition of the reduction s' of s has the consequence that all confluences and restriction equations are satisfied in Γ' at s'. We can conclude that s' does not only have the properties (a1), (a2) and (a3) of 2.7 but also the property (a4) with respect to Γ'. Therefore s' is a state for B'.

We now show that λ is a one-to-one mapping from the set of all states for B onto the set of all states for B'. For this purpose we have to prove that every state s' for B' has exactly one inverse image with respect to λ. Let s' be a state for B'. Let s' be the specification of values for all scaled variables, for all current and lagged tendencies and all system specific restrictions appearing in confluences and restriction equations of Γ which is obtained by complementing s by that value for ∂EV which results if the right hand side R of the confluence for ∂EV is evaluated at s'. Suppose that this s is not a state for B. Then there must be at least one confluence or restriction equation in Γ which is not satisfied at s. This cannot be the confluence for ∂EV, since the value of ∂EV at s has been constructed in such a way that the confluence for ∂EV is satisfied.

Let T be the main term of a confluence or restriction equation which is not satisfied. Let T_0 be the result of substituting ∂EV by the right hand side R of the confluence for ∂EV in T (see 6.3). At s' this expression T_0 has the same value as T at s. Since the four transformations 1, 2, 3, and 4 are equivalent transformations the value of the reduction T' of T at s' also coincides with the value of T at s. A boundary or system specific restriction has the same value at s and s'. Therefore the value of the right hand side of the confluence with the main term T has the same value at s as the right hand side of its reduction at s'. By assumption the confluence or restriction equation of Γ under consideration is not satisfied. Therefore its reduction is not satisfied contrary to the assumption that s' is a state for (Λ', Γ'). This shows that s must be a state of (Λ, Γ). Consequently, s is an inverse image of s' with respect to the state mapping λ.

It can be seen as follows that s is the only inverse image of s' with respect to λ. By the definition of λ the states s and s' specify the same values for all scaled variables, all current and lagged tendencies with the exception of ∂EV and all system specific restrictions. Moreover, the value of ∂EV at s is uniquely determined since it is the value of R at s'. Here it is of importance that EV is monocausal. This completes the proof of the lemma. □

Remark. *The proof has shown that for every XY different from EV the right hand side of the confluence for ∂XY or the restriction equation for $\Box XY$, if there is one, has the same value at s as the right hand side of its reduction at s'.*

Lemma 27. *Let EV be an eliminable variable of a base $B = (\Lambda, \Gamma)$ and let $B' = (\Lambda', \Gamma')$ be the reduction of B after the elimination of EV. Moreover let λ be the state mapping for the elimination of EV in B and let ω be a main transition cause, a halfway switch or a perturbance of a tendency ∂XY other than ∂EV. Then ω is invariant under the state mapping λ.*

Proof. We first show that the assertion holds for main transition causes and halfway switches. It has been pointed out at the beginning of this section that a shift of EV, a lag extinction of ∂EV^- or a tendency switch of ∂EV cannot be pending at a state of B. Consequently all main transition causes and halfway switches pending at a state s of B concern values of scaled variables in Λ' or values of current or lagged tendencies of variables in Λ'.

Each scaled variable and each current or lagged tendency of a variable in Λ' has the same value at s and $s' = \lambda(s)$. Therefore the same shifts and lag extinctions are pending at s and s'. In view of the remark after the proof of lemma 26 a tendency switch of a tendency ∂XY is pending at s in B, if and only if it is pending at s' in B'. The same is true for halfway switches. Consequently, the assertion holds for main transition causes and halfway switches.

It remains to show that the assertion holds as far as perturbances are concerned. Let $\omega = [\partial XY : d]$ be a perturbance of a tendency ∂XY other than ∂EV pending at a potentially stationary state s of B. Then at s the value of ∂XY is zero and d is in the value of the boundary restriction or system specific restriction for ∂XY, if there is one (see 3.3). A state s is potentially stationary if no shifts, no lag extinctions and no immediate tendency switches are pending at s. Since the assertion holds for main transition causes, it follows that $s' = \lambda(s)$ is potentially stationary in B' if and only if s is potentially stationary in B. In view of the remark after the proof of lemma 26 we can conclude that the perturbance $\omega = [\partial XY : d]$ is pending at $s' = \lambda(s)$ in B', if and only if it is pending at s in B. This completes the proof of the lemma. □

Remark. *The proof has shown that the state $s' = \lambda(s)$ is potentially stationary in B' if and only if s is potentially stationary in B.*

6.5. Reduction and modification

6.5.1. The reduced system. Let RV be a removable variable of the qualitative dynamic system

$$\Phi = (\Lambda, \Gamma, \rho, \alpha)$$

and let $B' = (\Lambda', \Gamma')$ be the reduction of its base $B = (\Lambda, \Gamma)$ after the elimination of RV. In the following it will be our aim to complement B' by a reduced priority ranking ρ' and a reduced perturbance assignment α' in order to define a "reduced system"

$$\Phi' = (\Lambda', \Gamma', \rho', \alpha')$$

and to show that Φ' is a qualitative dynamic system as defined in 3.7.

Let λ be the state mapping for the elimination of RV in B. The inverse of λ is denoted by λ^{-1}. The **reduced priority ranking ρ' after the elimination of RV in Φ** is defined by

$$\rho'(\omega, s') = \rho(\omega, \lambda^{-1}(s))$$

for every state s' of B' and for every main transition cause ω pending at s'.

Since fleeting, lasting, and exposed states are defined in terms of the main transition states pending at them, it follows by lemma 27 that ρ' satisfies conditions (d1), (d2), and (d3) in 3.5. Therefore ρ' is a priority ranking for B'. Moreover, in view of the remark after the proof of lemma 27 it is clear that a state $s' = \lambda(s)$ is potentially stationary in B' if and only if s is potentially stationary in B.

The **reduced perturbance assignment** α' **after the elimination of** RV assigns the set $\alpha'(s')$ of all perturbances ω with

$$\omega \in \alpha(\lambda^{-1}(s))$$

to every potentially stationary state s' of B'. Since RV satisfies the removability condition (e8), no perturbance of ∂RV is in the expected perturbance set $\alpha(s)$ of $s = \lambda^{-1}(s')$. It follows by lemma 27 that all perturbances pending at s' are also pending at s. Moreover in view of the remark after the proof of lemma 27 it is clear that a state $s' = \lambda(s)$ is potentially stationary in B' if and only if it is potentially stationary in B. We can conclude that α' has the properties of a perturbance assignment as defined in 3.6.

We can conclude that the reduced base (Λ', Γ') together with the reduced priority ranking ρ' and the reduced perturbance assignment α' after the elimination of RV form a qualitative dynamic system

$$\Phi' = (\Lambda', \Gamma', \rho', \alpha')$$

as defined in 3.7. This system Φ' is the **reduced system of** Φ **after the elimination of** RV or more shortly the **reduction of** Φ' **after the elimination of** RV.

6.5.2. Operators. Let EV be an eliminable variable in the base $B = (\Lambda, \Gamma)$ and let $B' = (\Lambda', \Gamma')$ be the reduction of B after the elimination of B. We use the notation

$$B' = M_{EV}(B)$$

in order to express the relationship between B and B'. We call M_{EV} the **elimination operator** for EV. This elimination operator is **applicable** to every base in which EV is an eliminable variable.

The common name **modifier** is used for tendency switches, halfway switches and perturbances of a tendency ∂XY in a base $B = (\Lambda, \Gamma)$. A modifier ω gives rise to a **modified base** $B_\omega = (\Lambda, \Gamma_\omega)$. In the case $\omega = [\partial XY \to d]$ of a tendency switch or a halfway switch, B_ω is the hypothetical base of B for ω and in the case $\omega = [\partial XY : d]$ of a perturbance, B_ω is the auxiliary base of B for ω. The relationship between B and B_ω is expressed by the notation

$$B_\omega = M_\omega(B)$$

We call M_ω the **modification operator** for ω. The modification operator M_ω with $\omega = [\partial XY \to d]$ or $[\partial XY : d]$ is **applicable** to every base $B = (\Lambda, \Gamma)$ with the following two properties:

(i) The variable XY is in Λ
(ii) The base B has at least one state at which ω is pending

As has been pointed out in 3.2.3 and 3.3.1 the modified base does not depend on the state or the states of B at which ω is pending. The modified base B_ω is well defined, even if ω is not pending at any state of B. However, there is no necessity to look at B_ω unless B has at least one state at which ω is pending.

A modification operator M_ω with $\omega = [\partial XY \to d]$ or $\omega = [\partial XY : d]$ is **compatible** with the elimination operator M_{EV}, if XY and EV are different variables. Otherwise M_ω and M_{EV} are **incompatible**. Suppose that M_ω and M_{EV} are compatible and applicable to B. It can be seen without difficulty that EV is eliminable in $M_\omega(B)$ and that therefore M_{EV} is applicable to $M_\omega(B)$. It follows by lemma 27 that ω is pending at the state $s' = \lambda(s)$ of $B' = M_{EV}(B)$ if ω is pending at the state s of B. Therefore M_ω is applicable to $M_{EV}(B)$. We say that the operators M_{EV} and M_ω **commute** if we have

$$M_\omega(M_{EV}(B)) = M_{EV}(M_\omega(B)).$$

The validity of this equation is the content of the following lemma 28.

LEMMA 28. *Let EV be an eliminable variable in a base $B = (\Lambda, \Gamma)$ and let $\omega = [\partial XY \to d]$ or $\omega = [\partial XY : d]$ be a modifier such that ω is pending at at least one state of B and M_ω is compatible with M_{EV}. Then M_ω is applicable to $M_{EV}(B)$ and M_{EV} is applicable to $M_\omega(B)$. Moreover we have:*

$$M_\omega(M_{EV}(B)) = M_{EV}(M_\omega(B))$$

In other words, the operators M_ω and M_{EV} commute.

PROOF. Just before the statement of the lemma it has been pointed out that M_ω is applicable to $M_{EV}(B)$ and M_{EV} is applicable to $M_\omega(B)$. It remains to show that M_ω and M_{EV} commute. Let VW be a variable different from EV and XY in B and let S be the main term of the restriction equation for $\Box VW$ or the confluence for ∂VW. The operator M_ω leaves S unchanged. The operator M_{EV} changes S to the same term S', regardless of whether M_{EV} is applied to B or to $M_\omega(B)$. Therefore the main term S is replaced by this term S' in $M_\omega(M_{EV}(B))$ as well as in $M_{EV}(M_\omega(B))$. The same argument also applies to the main term of the restriction equation for $\Box XY$. It remains to be shown that the main term of the confluence for ∂XY is the same one in $M_\omega(M_{EV}(B))$ and $M_{EV}(M_\omega(B))$.

Consider the case $\omega = [\partial XY \to d]$ of a tendency switch or a halfway switch of ∂XY. In this case M_ω replaces the right hand side of the confluence for ∂XY by d. Obviously the end result is the same one, regardless of whether first M_{EV} is applied to B and then B_ω or whether the two operators are applied in the reverse order. It is clear that in this case the two operators commute.

We now look at the remaining case. Let $\omega = [\partial XY : d]$ with $d \neq 0$ be a perturbance of ∂XY and let T be the main term of the confluence for ∂XY in B.

This main term depends on the combination of values of the scaled variables in B. In the following we look at T for a fixed but arbitrary combination of this kind.

Suppose that T has no variable components of the form ∂EV or $-\partial EV$. Then T is not changed by the application of M_{EV}. Regardless of whether M_ω is applied first and M_{EV} second or whether the two operators are applied in the reverse order, the end result is the same simplification T_ω of $T + d$. In this case the two operators commute. In the following we shall assume that T has at least one component of the form ∂EV or $-\partial EV$. Let W be the sum of all components of T of the form ∂EV or $-\partial EV$. Let V be the sum of all other variable components of T. Moreover let C be the constant component of T. In view of (c6) in 2.8 we cannot have $C = \{0\}$, since T has the variable components in C. Therefore either $C = d$ or $C = -d$ or $C = \{-, 0, +\}$ holds. However, the case $C = \{-, 0, +\}$ is excluded by (c7) in 2.8, since T has the variable components in W. Therefore we either have $C = d$ or $C = -d$.

Since M_ω is applicable to B, the base B has a potentially stationary state s at which ω is pending. At this state s the value of T must be zero in view of condition (i) for the perturbability of ∂XY at s. Consider the case that the right hand side of the confluence for ∂EV is a constant direction. Then the value of T at s cannot be zero unless the confluence for ∂EV has the form

$$\partial EV = 0$$

Another constant direction on the right hand side would result in a value of T at s different from zero.

Tables 39, 40 and 41 are based on a case distinction between $\partial EV = 0$ and all other possible forms of the confluence for ∂EV. This case distinction concerns the rows of the tables. In the case $\partial EV = 0$ it is important whether there are variable components different from ∂EV or $-\partial EV$ in T. These cases are shown in the columns of the tables. However, if the form of the confluence for ∂EV is different from $\partial EV = 0$ this case distinction with respect to the form of T is not necessary.

It has been shown that a constant component of T, if there is one, must have the value d or $-d$. Of course, T may not have any constant component. Each of the tables 39, 40 and 41 deals with one of the three possibilities. It is clear that the tables cover all cases which are still open. Since no distinction with respect to the structure of T has to be made, unless we have $\partial EV = 0$, the table has only three fields. The entries in these fields are figures which show, how T is changed by a successive application of M_{EV} and M_ω to B in this order or in the reverse one. It will now be argued that the figures correctly describe these changes.

Forms of the confluence for ∂EV	1) $T = V + W$	2) $T = W$
$\partial EV = 0$	1 $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & V \\ \downarrow M_\omega & & \downarrow M_\omega \\ T+d & \xrightarrow{M_{EV}} & V+d \end{array}$	2 $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & 0 \\ \downarrow M_\omega & & \downarrow M_\omega \\ T+d & \xrightarrow{M_{EV}} & d \end{array}$
$\partial EV = \partial TU$ $\partial EV = -\partial TU$ $\partial EV = \partial TU^-$ $\partial EV = -\partial TU^-$	3 $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & T' \\ \downarrow M_\omega & & \downarrow M_\omega \\ T+d & \xrightarrow{M_{EV}} & T'+d \end{array}$	

TABLE 39. The case of a main term T without constant components

1) The main term T has variable components other than ∂EV or $-\partial EV$. The sum of these components is V. The sum of the components of the form ∂EV or $-\partial EV$ is W

2) All variable components of T have the form ∂EV or $-\partial EV$

For easy reference the fields in the three tables are numbered from 1 to 9 in the upper right corner. In all three fields of Table 39, case 1 of Table 12 in 3.3.1 yields the conclusion that the application of M_ω to B changes the term T to $T+d$. In fields 1 and 2 the application of M_{EV} to B removes W and thereby changes T to V and 0, respectively. In field 3 the application of M_{EV} to B substitutes ∂EV by the right hand side of the confluence for ∂EV. Then duplicates are removed if necessary. The same steps are taken for $T + d$ in the application of M_{EV} to $B_\omega = M_\omega(B)$. Here, too we receive the same expression $T' + d$ in $M_{EV}(B_\omega)$ and $M_\omega(B')$. It follows that in all three cases of Table 39 the operators M_{EV} and M_ω commute.

Forms of the confluence for ∂EV	$T = d + V + W$	$T = d + W$
$\partial EV = 0$	[4] $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & d+V \\ \downarrow M_{\omega} & & \downarrow M_{\omega} \\ T & \xrightarrow{M_{EV}} & d+V \end{array}$	[5] $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & d \\ \downarrow M_{\omega} & & \downarrow M_{\omega} \\ T & \xrightarrow{M_{EV}} & d \end{array}$
$\partial EV = \partial TU$ $\partial EV = -\partial TU$ $\partial EV = \partial TU^-$ $\partial EV = -\partial TU^-$	[6] $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & T' \\ \downarrow M_{\omega} & & \downarrow M_{\omega} \\ T & \xrightarrow{M_{EV}} & T' \end{array}$	

TABLE 40. The case of d as the constant component of T *)

*) See the footnotes 1) and 2) below Table 39

In Table 40 the main term T is not changed by M_ω in view of $d + d = d$. Therefore we have $B = B_\omega(B)$. Obviously, M_{EV} and M_ω commute in the three cases of Table 40.

In all three cases of Table 41 the main term of T contains $-d$. This is not changed by the application of M_{EV}. Therefore in $M_\omega(B)$ as well as in $M_\omega(M_{EV}(B))$ the term T is changed to $\{-, 0, +\}$. Obviously, the application of M_{EV} to $M_\omega(B)$ does not involve further changes of T. We can conclude that M_{EV} and M_ω commute in the three cases of Table 41. This completes the proof of the lemma.

□

COMMENT. *It is the task of this chapter to show that the elimination of a removable variable does not change any important feature of the system. Since by (e8) a removable variable is perturbance free, it follows by lemma 27, that transition causes are invariant under the elimination of a removable variable. Therefore*

Forms of the confluence for ∂EV	$T = -d + V + W$	$T = -d + W$
$\partial EV = 0$	7 $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & -d+V \\ \downarrow M_\omega & & \downarrow M_\omega \\ \{-,0,+\} & \xrightarrow{M_{EV}} & \{-,0,+\} \end{array}$	8 $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & -d \\ \downarrow M_\omega & & \downarrow M_\omega \\ \{-,0,+\} & \xrightarrow{M_{EV}} & \{-,0,+\} \end{array}$
$\partial EV = \partial TU$ $\partial EV = -\partial TU$ $\partial EV = \partial TU^-$ $\partial EV = -\partial TU^-$	9 $\begin{array}{ccc} T & \xrightarrow{M_{EV}} & T' \\ \downarrow M_\omega & & \downarrow M_\omega \\ \{-,0,+\} & \xrightarrow{M_{EV}} & \{-,0,+\} \end{array}$	

TABLE 41. The case of $-d$ as the constant component of T [*)]

[*)] See the footnotes 1) and 2) below Table 39

lemma 27 was an important step towards the goal of this chapter. However the invariance of transition causes is not enough. A similar invariance property has to be derived for transition results. For this purpose it is necessary to look at realizations of the readjustment processes not only in a base B and its reduction B' but also in the hypothetical and auxiliary bases of B and B'. For this purpose lemma 28 will be of crucial importance. More about this will be said in 6.7.2.

6.6. The prestate mapping

Let EV be an eliminable variable of a base $B = (\Lambda, \Gamma)$ and let $B' = (\Lambda', \Gamma')$ be the reduction of B after the elimination of EV. In 4.1 the notion of a prestate has been introduced. A prestate differs from a state in three ways. Instead of one value for a current tendency a prestate specifies two values for a left and a right tendency. A prestate specifies a confirmation status L or F for every directional. Confluences and restriction equations need not be satisfied at a prestate.

If ω is a modifier, then the space of prestates is the same for B and $B_\omega = M_\omega(B)$. A prestate p' for the reduction B' of B differs from a prestate p for B only inasmuch as no values for ∂EV_L and ∂EV_R and no comfirmation status of EV is specified by p'.

For every prestate p of B an **associated** prestate $p' = \pi(p)$ for B' is defined as follows: p' results from p by taking out the specifications of ∂EV_L and ∂EV_R and the confirmation status of ∂EV but leaving everything else unchanged. The function π which connects the prestates of B to their associated prestates of B' is called the **prestate mapping for the elimination of EV in B**.

Unlike the state mapping defined in 6.4, the prestate mapping is not a one-to-one mapping. It is a mapping from the set of all prestates in B onto the set of all states of B'. However, a state of B' does not have a unique inverse image under this mapping. For every prestate p' of B' let $\pi^{-1}(p')$ be the set of all p with $p' = \pi(p)$. There are 18 ways of specifiying ∂EV_L, ∂EV_R and the confirmation status of ∂EV. Therefore $\pi^{-1}(p')$ has 18 elements.

As has been explained in 4.3 a start is a prestate p_0 with $\partial XY_L = \partial XY_R$ for every tendency ∂XY and with the additional property that at p_0 every confirmation status has the value L. A start p_0 for B is called **EV-adjusted**, if at p_0 the confluence for EV is satisfied.

A start is not necessarily EV-adjusted. Consider the example of an eliminable variable EV with the confluence

$$\partial EV = \partial XY^-$$

Let ω be the lag extinction of ∂XY^- pending at a state s. Then at $p_0(\omega, s)$ the left and right tendencies of ∂EV have the same value as ∂EV at s, but the value of ∂XY^- is changed. Obviously the confluence for ∂EV is not satisfied at $p_0(\omega, s)$.

In 4.3 the prestate $p_0(s)$ of a state s for B has been defined. The prestate $p'_0(s')$ of state s' for B' is defined analogously as the prestate for B' at which scaled variables, lagged tendencies, and system specific restrictions have the same values as at s', at which the value of a left and right tendency ∂XY_L and ∂XY_R is the value of ∂XY at s', and at which every confirmation status is L. The notation $p'_0(\omega, s')$ is used for the transition start of ω at a state s' of B' at which ω is pending.

LEMMA 29. *Let EV be an eliminable variable of a base $B = (\Lambda, \Gamma)$ and let $B' = (\Lambda', \Gamma')$ be the reduction of B after the elimination of EV. Moreover, let λ be the state mapping and π be the prestate mapping for the elimination of EV in B. Then the following statements are true:*

(1) *The prestate mapping is a mapping from the set of all prestates of B onto the set of all prestates of B'.*

(2) *Let s be a state of B and let ω be a transition cause other than a perturbance of ∂EV or let ω be a halfway switch pending at s. Moreover let $s' = \lambda(s)$ be the image of s under the state mapping λ. Then we have*

$$p_0'(s') = \pi(p_0(s))$$

and

$$p_0'(\omega, s') = \pi(p_0(\omega, s))$$

(3) *For every state s of B the prestate $p_0(s)$ is EV-adjusted.*

(4) *If ω is a transition cause or a halfway switch pending at a state s of B, such that $p_0(\omega, s)$ is not EV-adjusted then ω is a lag extinction of a lagged tendency ∂XY^- and the confluence for ∂EV either has the form $\partial EV = \partial XY^-$ or$\partial EV = -\partial XY^-$.*

Proof. Statement (1) repeats a conclusion reached immediately after the definition of the prestate mapping. We now look at statement (2). The state $s' = \lambda(s)$ is obtained from s by leaving out the specification of ∂EV and leaving everything else unchanged. The prestate $p_0(s)$ results from s by attaching the value of ∂XY in s to ∂XY_L and ∂XY_R in $p_0(s)$ for every tendency ∂XY in B, by leaving the values of all other components of s unchanged and by specifying the confirmation status of every directional as L. The way in which $p_0'(s')$ is connected to s' is analogous. Therefore $p_0'(s')$ can be obtained from $p_0(s)$ by leaving out the specifications of ∂EV_L and ∂EV_R and by changing nothing else, or in other words as $\pi(p_0(s))$. This shows that the first equation of (2) holds.

If ω is a tendency switch, a halfway switch or a perturbance, then $p(\omega, s)$ is nothing else than $p_0(s)$. Therefore in this case the second equation is an immediate consequence of the first one. If ω is a shift of a scaled variable XY to a new value then $p_0(\omega, s)$ differs from $p_0(s)$ only by this new value of XY and the same is true for $p_0'(\omega, s')$ and $p'(s')$. Therefore leaving out the specifications of ∂EV_L and ∂EV_R in $p_0(\omega, s)$ yields $p_0'(\omega, s')$. This shows that the second equation holds in the case of a shift. The case of a lag extinction is analogous. Consequently (2) holds.

We now turn our attention to (3). Since s is a state the confluence for ∂EV is satisfied at s. Therefore this confluence is also satisfied at $p_0(s)$. Consequently (3) holds.

Finally we look at (4). In view of (3) nothing has to be proved for the cases in which $p_0(\omega, s) = p_0(s)$ holds. In the case of a shift of a scaled variable XY the prestate $p_0(\omega, s)$ differs from $p_0(s)$ only by the value of XY. Since the confluence for ∂EV is scale independent, it follows by (3) that in this case $p_0(\omega, s)$ is EV-adjusted.

Now assume that ω is the lag extinction of a lagged tendency ∂XY^- and that the confluence for ∂EV neither has the form $\partial EV = \partial XY^-$ nor $\partial EV = -\partial XY^-$.

This case is similar to that of a shift. $p_0(\omega, s)$ differs from $p_0(s)$ only by the value of ∂XY^- and this lagged tendency does not appear on the right hand side of the confluence for ∂EV. Therefore $p_0(\omega, s)$ is EV-adjusted in this case, too. This completes the proof of the lemma. □

6.7. Reducibility

6.7.1. Invariance of transition results. As in the preceding section let EV be an eliminable variable of a base $B = (\Lambda, \Gamma)$ and let $B' = (\Lambda', \Gamma')$ be the reduction of B after the elimination of EV. Moreover let λ be the state mapping and π be the prestate mapping for the elimination of EV in B.

Transition causes and halfway switches other than perturbances of ∂EV are invariant under the state mapping. This was the content of lemma 27. In the following a definition of invariance under the state mapping of the result of a transition cause ω at a state s will be given. Later in this chapter it will be shown that a transition cause ω pending at a state s of B which is not a perturbance of ∂EV always has the property that the result of ω at s is invariant under the state mapping λ in the sense of this definition. This will be important for the conclusion that the elimination of an eliminable variable does not lead to an essential change of the transition diagram and the extended transition diagram.

In 5.2 a readjustment result $h(\omega, s)$ and a transition result $z(\omega, s)$ have been defined for every realizable main transition cause ω pending at a state s of B. The only main transition causes which are not realizable are infeasible tendency switches. The readjustment result $h(\omega, s)$ is the final prestate of a realization of the readjustment process in B starting with $p_0(\omega, s)$ if ω is a reanchoring, i.e., a shift or a lag extinction. If ω is a feasible tendency switch then $h(\omega, s)$ is the final prestate of a readjustment process in the hypothetical base $B_\omega = M_\omega(B)$ starting with $p_0(\omega, s)$. If $\omega = [\partial XY \to d]$ is a semifeasible tendency switch, then $h(\omega, s)$ is the final prestate of a readjustment process in the hypothetical base B_μ for the halfway switch $\mu = [\partial XY \to 0]$ starting with $p_0(\mu, s)$. The transition result $z(\omega, s)$ is the state generated by $h(\omega, s)$. Transition results remain undefined for infeasible tendency switches.

Consider a realizable main transition cause ω at a state s' of B'. We use the notation $h'(\omega, s')$ for the readjustment result and $z'(\omega, s')$ for the transition result of ω at s' in B'. We say that the **result of ω at s is invariant under the** state mapping if we have

$$z'(\omega, s') = \lambda(z(\omega, s)) \quad \text{for} \quad s' = \lambda(s)$$

(The definitions of readjustment results and transition results have been repeated here, in order to emphasize their somewhat involved meaning in the case of a tendency switch.)

Let $\omega = [\partial XY : d]$ be a perturbance pending at a potentially stationary state s of B and let $s' = \lambda(s)$ be the image of s under the state mapping. The set of all reentry states e which can be reached after the perturbance ω at s is denoted by $E(\omega, s)$. Consider a system $\Phi = (\Lambda, \Gamma, \rho, \alpha)$ with the base B. Obviously, the reentry state set $E(s)$ defined in 5.8 is the union of all $E(\omega, s)$ with $\omega \in \alpha(s)$. We call $E(\omega, s)$ the **reentry state set after ω at s.**

In view of the remark after the proof of lemma 26 the state $s' = \lambda(s)$ is potentially stationary in B'. The **reentry set after** ω at s' is defined analogously to $E(\omega, s)$ and is denoted by $E'(\omega, s')$.

Assume that XY and EV are different variables. We say that the **result of ω at s is invariant under the state mapping**, if $E'(\omega, s')$ is the set of all $e' = \lambda(e)$ with $e \in E(\omega, s)$.

Finally the definition of invariance under the state mapping is extended to the case that ω is an infeasible tendency switch at a state s of B. In this case we say that the **result of ω at s is invariant under the state mapping** if ω is infeasible at $s' = \lambda(s)$. This way of speaking involves a slight abuse of language, since we take the point of view that an infeasible tendency switch does not cause a transition and therefore does not lead to a transition result. Nevertheless this way of completing the definition of invariance under the state mapping of ω at s is convenient and seems to be natural.

6.7.2. Preview. It is the aim of this chapter to show that the elimination of a removable variable does not involve essential changes of the transition diagram or the extended transition diagram. In these diagrams for a system Φ a node represents a state s. In the reduction Φ' of Φ after the elimination of a removable variable RV of Φ the same node represents the image $s' = \lambda(s)$ under the state mapping for the elimination of RV. Everything else remains unchanged. The edges are associated to the same transition causes. In order to show this, it is necessary to derive "invariance of transition results" in the following sense: If EV is an eliminable variable and ω is a transition cause other than a perturbance of ∂EV, pending at state s, then ω at s is invariant under the state mapping for the elimination of EV.

The way towards the establishment of invariance of transition results will be long and tedious. A tool for tackling this task will be presented in 6.7.3. Consider a pair of realizations $p_0, \ldots, p_N$ in a base B and $p'_0, \ldots, p'_L$ in the reduction B' of B after the elimination of an eliminable variable EV. Assume that p_0 and p'_0 are the transition starts $p_0 = p_0(\omega, s)$ and $p'_0 = p'_0(\omega, s')$ for a transition cause ω

pending at s and $s' = \lambda(s)$, respectively. It is necessary to prove that p'_L is the image $\pi(p_N)$ of p_N under the prestate mapping for the elimination of EV in B.

It follows by theorem 3 on the order independence of the final prestate of a readjustment process that for the purpose of proving $p'_L = \pi(p_N)$ one can restrict one's attention to special realizations. The realization $p_0, \ldots, p_N$ will be assumed to be "EV-reducible". This means that it has certain properties which permit the construction of a special realization $p'_0, \ldots, p'_L$ in B'. This construction proceeds as follows: One first forms a "preliminary reduction" $p^0_0, \ldots, p^0_N$ with $p^0_k = \pi(p_k)$ for $k = 0, \ldots, N$ and then the "reduction" by leaving out the p^0_k with the property that the value or the confirmation status of ∂EV is changed in the step from p_{k-1} to p_k. The "EV-reduction" will be shown to be a realization of the readjustment process in B'.

Two cases need to be distinguished with respect to the definition of an EV-reducible realization: The link case in which EV is a link and the source case in which EV is a source. The definition is quite simple in the source case but considerably more complex in the link case.

It has to be shown that an EV-reducible realization always exists. This will be done in 6.8. In 6.9 it will be shown that the EV-reduction is a realization of the readjustment process in B'. We refer to this as the "realization property".

The tool of EV-reducible realizations and their EV reductions can be directly applied to the case of a shift or a lag extinction. Here invariance of transition results concerns readjustment processes in a base B and in its reduction B' after the elimination of EV. In the case of a tendency switch ω one has to look at realizations of the readjustment process in the hypothetical bases $B_\omega = M_\omega(B)$ and $B'_\omega = M_\omega(M_{EV}(B))$. Here it is of crucial importance that in view of lemma 28 the base B'_ω is also the reduction of B_ω after the elimination of EV. This makes it possible to apply the tool of EV-reducible realizations and their EV-reductions to B_ω and B'_ω.

If a tendency switch $\omega = [\partial XY \to d]$ turns out not to be feasible one has to look at readjustment processes in the hypothetical bases $B_\mu = M_\mu(B)$ and $B'_\mu = M_\mu(M_{EV}(B))$ where μ is the halfway switch $\mu = [\partial XY \to 0]$. Here, too, lemma 28 permits the application of the tool of EV-reducible realizations to B_μ and B'_μ. In this way it is possible to show that invariance of transition results holds for main transition causes.

Consider the case of a perturbance ω of a tendency other than ∂EV pending at a potentially stationary state s of B. In this case lemma 28 together with the invariance of main transition causes can be applied to the auxiliary bases B_ω of B and B'_ω of B'. In this way it will be possible to construct a one-to-one relationship between the reentry histories after ω in B and B', which permits the conclusion that the invariance of transition results holds for ω at s.

6.7.3. EV-reducible realizations. Let EV, B, B', λ and π be defined as in 6.7.1. As we have seen in 6.2.1 an eliminable variable is either a source or a link. A case distinction will be made between the **source case**, in which EV is a source and the **link case** in which EV is a link. The definition of an RV-reducible realization in B is straightforward in the source case and somewhat less simple in the link case.

Let p_0 be a start for B. In the source case a realization $p_0, \ldots, p_N$ of the readjustment process in B is called EV-**reducible** if ∂EV is adapted and confirmed in the step from p_0 to p_1. It will be part of the content of lemma 30 that in the source case an EV-reducible realization $p_0, \ldots, p_N$ of the readjustment process in B exists for every arbitrary start p_0 for B. This is important in view of the exceptional case in statement (4) of lemma 29.

In the link case a transition start is always EV-adjusted. Assume that EV is a link with ∂XY or $-\partial XY$ as the determinator of ∂EV. In this case a realization $p_0, \ldots, p_N$ of the readjustment process in B is called EV-reducible, if the following four conditions are satisfied.

- (g1) The start p_0 is EV-adjusted
- (g2) If and only if ∂XY is confirmed or adapted and confirmed in the step from p_{j-1} to p_j, the tendency ∂EV is confirmed or adapted and confirmed in the step from p_j to p_{j+1}. This is true for $j = 1, \ldots, N-1$. (Not necessarily the same activity is applied to ∂XY and ∂EV.)
- (g3) If and only if ∂XY is dampened in the step from p_{j-1} to p_j, the tendency ∂EV is dampened in the step from p_j to p_{j+1}. This is true for $j = 1, \ldots, N-1$.
- (g4) If activity 3 is applied to ∂EV then ∂EV has been dampened before and ∂EV is adapted in the first step of the adaptation phase immediately following the dampening phase. (This is the step from $r(4,1)$ to the next prestate in $p_0, \ldots, p_N$.) Moreover ∂XY is adapted after ∂EV in the same adaptation phase.

Conditions (g2) and (g3) are understood as implying that the concerning operations cannot be applied to ∂EV in the step from p_0 to p_1 and to ∂XY in the step from p_{N-1} to p_N, since there is no p_{j-1} for $j = 0$ and no p_{j+1} for $j = N$.

In the source case as well as in the link case, a prestate p_j in an EV-reducible realization $p_0, \ldots, p_N$ in B is called **exceptional**, if an operation is applied to ∂EV in the step from p_j to p_{j+1}. This step is then also called **exceptional**. This definition has to be understood as implying that p_N cannot be exceptional, since there is no step from p_N to p_{N+1}. In the source case p_0 is the only exceptional prestate. In the link case there may be more than one exceptional prestate. A prestate p_j in an EV-reducible realization is called **normal** if it is not exceptional.

In the following the notion of an EV-reduction will be introduced. Here no distinction is made between the source case and the link case.

The EV-**reduction** $p'_0, \dots, p'_L$ of an EV-**reducible** realization $p_0, \dots, p_N$ is obtained as follows: First a **preliminary** EV-**reduction** $p^0_0, \dots, p^0_N$ with

$$p^0_j = \pi(p_j) \quad \text{for} \quad j = 0, \dots, N$$

is formed then all prestates $p^0_j = \pi(p_j)$ are taken out of $p^0_0, \dots, p^0_N$, for which p_j is exceptional. The remaining prestates of $p^0_0, \dots, p^0_N$ are then renumbered consecutively from 1 to L. In this way one obtains the EV-reduction $p'_0, \dots, p'_L$ of $p_0, \dots, p_N$.

In the EV-reduction the index j of p^0_j is replaced by a new index $m = \tau(j)$. We call τ the **renumbering function**. Neither in the source case nor in the link case the last prestate p_N of $p_0, \dots, p_N$ can be exceptional. In the source case only p_0 is exceptional and there must be at least one other prestate p_1 in $p_0, \dots, p_N$. For the reasons explained above p_N cannot be exceptional. In both cases we have $L = \tau(N)$.

6.8. Existence of EV-reducible realizations

In the following two lemmas will be proved. Theorem 7 asserts the existence of EV-reducible realizations for arbitrary starts in the source case and for EV-adjusted starts in the link case. Lemma 31 has the purpose to make it clear that all cases of starts arising in the theory proposed here are covered by lemma 30.

THEOREM 7. *Let $B = (\Lambda, \Gamma)$ be a base and let EV be an eliminable variable in B. Moreover let p_0 be a start for B. Assume that one of the following two conditions 1) and 2) is satisfied:*

1) *EV is a source.*
2) *p_0 is EV-adjusted and EV is a link*

Then an EV-reducible realization $p_0, \dots, p_N$ of the readjustment process in B, starting with p_0 exists.

PROOF. Assume that EV is a source. Then ∂EV is mature at p_0. Therefore ∂EV can be adapted and confirmed in the step from p_0 to p_1. The sequence $p_0, \dots, p_N$ can then be continued in any way permitted by the definition of the readjustment process. Obiously one thereby receives an EV-reducible realization.

From now on we assume that EV is a link and that p_0 is EV-adjusted. Let ∂XY be the determinator of EV. As in section 6.3 the right hand side of the

	Form of confluence for ∂EV	
case	$\partial EV = \partial XY$	$\partial EV = -\partial XY$
1	$\partial EV_L = \partial XY_L \neq 0$ $\partial EV_R = \partial XY_R = 0$	$\partial EV_L = -\partial XY_L \neq 0$ $\partial EV_R = \partial XY_R = 0$
2	$\partial EV_L = \partial EV_R = \partial XY_L = \partial XY_R \neq 0$	$\partial EV_L = \partial EV_R = -\partial XY_L = -\partial XY_R \neq 0$
3	$\partial EV_L = \partial EV_R = \partial XY_L = \partial XY_R = 0$	$\partial EV_L = \partial EV_R = -\partial XY_L = -\partial XY_R = 0$

TABLE 42. Cases at $r(4,1)$

confluence for ∂EV is denoted by R. The proof of the assertion will make use of the flow chart of figure 8 in 4.5. We shall follow the course of the readjustment process in order to construct an EV-reducible realization by taking advantage of the freedom of order, in which an activity is applied during a phase of its application.

We first look at the possibility that ∂XY is adapted and confirmed at rectangle 3 of figure 8. Obviously in this case ∂EV cannot become mature before ∂XY, but immediately after the application of activity 1 to ∂XY the tendency ∂EV is mature and adaptation and confirmation of ∂EV can follow immediately. Consequently a first phase the application of activity 1 can be arranged as required by (g2). After the adaptation and confirmation of ∂EV one obtains an EV-reducible realization by continuing in any permissible way.

Since the case of adaptation and confirmation of ∂XY and ∂EV at rectangle 3 has been clarified, it will be assumed in the following that at $r(2,1)$ the tendencies ∂XY and ∂EV are still loose. Up to $r(2,1)$ all changes of left and right tendencies concern tendencies which are firm at $r(2,1)$. Therefore at $r(2,1)$ the right and left tendencies of ∂XY and ∂EV have the same values as at the EV-adjusted start p_0. The tendencies ∂XY and ∂EV are univalued and the confluence for ∂EV is satisfied at $r(2,1)$. Either both of them are non-zero tendencies or both of them are zero-tendencies at $r(2,1)$.

At $r(2,1)$ a dampening phase may begin. This phase ends at $r(4,1)$. During a dampening phase only values of right tendencies are changed. We now look at the question what happens to ∂XY_R and ∂EV_R between $r(2,1)$ and $r(4,1)$. The values of the left and right tendencies of ∂XY and ∂EV remain unchanged if ∂XY and ∂EV are not dampened.

We distinguish 3 cases described by Table 42 which may arise at $r(4,1)$ with respect to the values of the right and left tendencies of ∂EV and ∂XY. At $r(2,1)$ these two tendencies are univalued and all their left and right tendencies are equal. In cases 1 and 2 ∂EV and ∂XY are non-zero tendencies and in case 3 they are zero-tendencies at $r(2,1)$ and therefore also at $r(4,1)$.

	Form of confluence for ∂EV	
case	$\partial EV = \partial XY$	$\partial EV = -\partial XY$
4	$\partial XY_L \neq 0$ $\partial EV_L = \partial EV_R = \partial XY_R = 0$	$\partial XY_L \neq 0$ $\partial EV_L = \partial EV_R = \partial XY_R = 0$
5	$\partial EV_L = \partial EV_R = \partial XY_L = \partial XY_R \neq 0$	$\partial EV_L = \partial EV_R = -\partial XY_L = -\partial XY_R \neq 0$
6	$\partial EV_L = \partial EV_R = \partial XY_L = \partial XY_R = 0$	$\partial EV_L = \partial EV_R = -\partial XY_L = -\partial XY_R = 0$

TABLE 43. Cases at $r(6,1)$

Case 1 arises if ∂XY and ∂EV are non-zero tendencies and ∂XY is maladjusted at $r(2,1)$ or becomes maladjusted between $r(2,1)$ and $r(4,1)$. As long as ∂XY is not dampened, ∂EV remains adjusted, but as soon as ∂XY is dampened ∂EV becomes maladjusted and has to be dampened, too. The realization can be built up in such a way that ∂EV is dampened immediately after ∂XY as required by (g3) in the definition of an EV-reducible realization for the link case.

In case 2 the tendency ∂XY is an adjusted non-zero tendency at $r(2,1)$ and does not become maladjusted during the dampening phase between $r(2,1)$ and $r(4,1)$. Neither ∂XY nor ∂EV is dampened in this case. The values of their right and left tendencies at $r(4,1)$ are the same ones as at $r(2,1)$ and therefore the same ones as at p_0.

Case 3 arises, if ∂XY and ∂EV are univalued zero tendencies at $r(2,1)$. Zero tendencies are not dampened. Therefore, in this case, too, the values of the left and right tendencies of ∂EV and ∂XY are the same ones as at $r(2,1)$ and $r(4,1)$.

At $r(4,1)$ a phase of activity 3 begins, if there are maladjusted tendencies at $r(4,1)$. Such a phase ends at $r(6,1)$. We now examine what happens between $r(4,1)$ and $r(6,1)$. It will become clear that the three cases 4, 5, and 6 shown by Table 43 can arise with respect to the left and right tendencies of ∂XY and ∂EV at $r(6,1)$. The three cases of Table 43 are numbered from 4 to 6 in order to avoid confusion with the cases 1 to 3 of Table 42.

From now on we shall assume that the confluence for ∂EV has the form $\partial EV = \partial XY$. Analagous arguments are valid for $\partial EV = -\partial XY$. In case 1 the tendency ∂XY is a non-zero tendency which has been dampened, because it was maladjusted and ∂EV has been dampened, too. At $r(4,1)$ the tendency ∂EV is maladjusted since we have $\partial EV \neq 0$ and $\partial XY_R = 0$. The tendency ∂XY is also maladjusted at $r(4,1)$ since it was maladjusted when it was dampened and since by lemma 10 a maladjusted tendency remains maladjusted when other tendencies are dampened. Therefore the realization can be built up in such a way that ∂EV is adapted in the step from $r(4,1)$ to the next prestate and ∂XY is adapted in the same adaptation phase as required by (g4). However the value of the right hand side of the confluence for ∂XY at $r(4,1)$ may be zero or the value of $-\partial XY_L$. If it is zero, then case 6 of Table 43 is obtained. The other possibility leads to case 4.

We now look at case 2 of Table 42. Here ∂XY and ∂EV have not been dampened and therefore are adjusted non-zero tendencies at $r(4,1)$. The right and left tendencies of ∂XY and ∂EV are not changed by activity 3. Therefore case 2 of Table 42 leads to case 5 of Table 43.

In case 3 of Table 42 the tendency ∂XY may be maladjusted at $r(4,1)$ and may have to be adapted to a value unequal to zero. However ∂EV remains adjusted in view of $\partial XY_R = 0$ if this happens. This leads to case 4 of Table 43. If ∂XY is adjusted at $r(4,1)$ then one arrives at case 6 of Table 43.

At $r(6,1)$ all tendencies are adjusted. In a phase of activity 4 beginning there, adjusted non-zero tendencies are confirmed. Confirmation of an adjusted tendency changes its confirmation status. In the case of a split tendency it also changes the value of the right tendency.

We now want to examine what happens between $r(6,1)$ and $r(8,1)$. Consider case 4 of Table 43. Here ∂XY is an adjusted split tendency. ∂EV becomes a maladjusted mature zero tendency as soon as ∂XY is confirmed. Obviously ∂EV cannot be confirmed between $r(6,1)$ and $r(8,1)$. Nevertheless the realization can be arranged as required by (g2). For this purpose one has to confirm ∂XY just before $r(8,1)$. This can be done, since in view of lemma 4 in 4.6 the order in which the adjusted non-zero tendencies are confirmed in a phase of activity 4 is arbitrary. In the step from $r(8,1)$ to the next prestate ∂EV can then be adapted and confirmed in a phase of activity 1. In this way one meets requirement (g2). After the confirmation of ∂EV the realization can be continued in any way compatible with the definition of the readjustment process. The conditions (g2), (g3), and (g4) concern applications of activities to ∂XY and ∂EV only, and therefore cannot be violated after ∂XY and ∂EV have beome firm.

Now consider case 5 of Table 43. In case 5 the tendencies ∂XY and ∂EV are adjusted non-zero tendencies and ∂EV can be confirmed immediately after ∂XY such that condition (g2) is met. From $r(8,1)$ the realization can be continued in any way compatible with the definition of the readjustment process.

In case 6 of Table 43 the left and right tendencies of ∂XY and ∂EV are all zero at $r(6,1)$ and therefore are not changed by the confirmation of non-zero tendencies between $r(6,1)$ and $r(8,1)$. At $r(8,1)$ a phase of activity 1 may begin. If ∂XY is adapted and confirmed there, then ∂EV becomes mature and can be adapted and confirmed immediately after ∂XY as required by (g2). The realization can then be continued in any way compatible with the definition of the readjustment process.

It remains to show what has to be done, if ∂XY and ∂EV are still loose at $r(10,1)$. From now on we shall assume that this is the case. At $r(10,1)$ the tendencies ∂XY and ∂EV are univalued zero tendencies. The tendency ∂EV is adjusted but ∂XY may be adjusted or not.

The construction of an EV-reducible realization will be continued in a recursive way. It will be shown that at $r(10,m)$, if it is reached, either ∂XY and ∂EV are firm or both tendencies are loose univalued zero tendencies. Since a realization ends after a finite number of steps, there must be a number $\overline{m}$ such that the answer to the question of switch 12 after $r(10,\overline{m})$ is NO. It will be discussed later what happens after $r(10,\overline{m})$. Let m be one of the numbers $1,\ldots,\overline{m}-1$. We proceed from the assumption that ∂XY and ∂EV are loose univalued zero tendencies at $r(10,1)$ and that $\overline{m}$ is greater than 1. We have to show that if ∂XY and ∂EV are loose univalued tendencies at $r(10,m)$ this is still true at $r(10,m+1)$ unless ∂XY and ∂EV are firm at $r(10,m+1)$.

In view of $m<\overline{m}$ the question of switch 12 is answered by YES at $r(10,m)$ At $r(10,m)$ a phase of activity 3 begins which lasts up to $r(6,m+1)$. Obviously ∂EV is adjusted at $r(10,m)$ and therefore is not adapted between $r(10,m)$ and $r(6,m+1)$ as required by (g4). The situation at $r(10,m)$ is similar to that at $r(4,1)$, but with the difference that now we must be in case 3 of Table 42. Therefore we come to case 4 or 6 of Table 43 at $r(6,m+1)$.

In case 6 the tendency ∂XY may become mature between $r(8,m+1)$ and $r(10,m+1)$ and if this happens ∂EV can be adapted and confirmed immediately after ∂XY as required by (g2), however, ∂XY and ∂EV are still loose at $r(10,m+1)$ then they are univalued zero tendencies there. Of course, whenever ∂XY and ∂EV have become firm, the realization can be continued in any way compatible with the definition of the readjustment process. We have shown what we wanted to show about $r(10,m+1)$ for case 6.

Now assume that at $r(6,m+1)$ the situation is like the one of case 4 of Table 43. Then ∂XY is an adjusted non-zero tendency at $r(6,m+1)$, but ∂EV is still an adjusted zero tendency. The realization can be arranged in such a way that ∂XY is confirmed at the end of the phase of activity 4 between $r(6,m+1)$ and $r(8,m+1)$. Then ∂EV can be adapted and confirmed in the step from $r(8,m+1)$ to the next prestate, as required by (g2). After this step the realization can be continued in any way compatible with the definition of the readjustment process.

Now suppose that at $r(10,\overline{m})$ the tendencies ∂XY and ∂EV are still loose univalued zero tendencies. In view of lemma 6 at $r(\overline{m},10)$ all loose tendencies are adjusted zero tendencies. A phase of activity 5 begins at $r(10,\overline{m})$, if there are any loose tendencies there. The order in which activity 5 is applied to loose tendencies is arbitrary. If ∂XY and ∂EV are still loose at $r(10,\overline{m})$, the tendency ∂EV can be confirmed immediately after ∂XY as required by (g2). We have shown how an EV-reducible realization can be constructed. □

LEMMA 30. *Let $B = (\Lambda, \Gamma)$ be a base and let EV be an eliminable variable in B. Then the following statements hold:*

(1) *Let ω be a reanchoring (a shift or a lag extinction) pending at a state s of B. Then an EV-reducible realization of the readjustment process running in B and starting with $p_0 = p_0(\omega, s)$ exists.*
(2) *Let ω be a modifier pending at a state s of B. Then an EV-reducible realization of the readjustment process running in the modified base $B_\omega = M_\omega(B)$ and starting with $p_0 = p_0(\omega, s)$ exists.*
(3) *Let ω be a perturbance of a tendency other than ∂EV pending at a state s of B and let a_M be a lasting state of the auxiliary base $B_\omega = M_\omega(B)$. Then an EV-reducible realization of the readjustment process running in B and beginning with the return start $q = p_0(a_M)$ exists. (See Table 26 in 5.8).*

PROOF. For the case that EV is a source each of the three statements is a direct consequence of Theorem 7. In the following we shall assume that EV is a link. Let ∂XY be the determinator of ∂EV. It follows by (4) in lemma 29 that $p_0(\omega, s)$ is EV-adjusted. Therefore the first statement holds.

Consider the case that ω is a tendency switch or a halfway switch of a current tendency pending at s. Obviously $p_0(\omega, s)$ is EV-adjusted in the hypothetical base B_ω if this current tendency is not ∂XY. If ω is a tendency switch or a halfway switch of ∂XY then EV is a source in B_ω. Therefore, the second statement holds in this case, too.

Now consider the case that ω is a perturbance pending at s. Then EV is a source in the auxiliary base $M_\omega(B)$. The second statement holds in this case, too.

It remains to prove the third statement. In the auxiliary system B_ω the confluence for ∂EV is the same one as in B. Since a_M is a state of B_ω the confluence for ∂EV is satisfied at a_M in B and consequently also at $p_0(a_M)$. This completes the proof of the Lemma. □

6.9. The realization property

6.9.1. Definitions and notational conventions. In the remainder of this chapter EV will always be an eliminable variable in a base $B = (\Lambda, \Gamma)$ and B' stands for the reduced base $M_{EV}(B)$. Moreover $p_0, \ldots, p_N$ will always be a fixed but arbitrary EV-reducible realization in B. Similarly $p_0^0, \ldots, p_N^0$ will be the preliminary EV-reduction and $p'_0, \ldots, p'_L$ will be the EV-reduction of $p_0, \ldots, p_N$. As before the symbol R will be used for the right hand side of the confluence for ∂EV

in B. The symbol λ will be used for the state mapping and π for the prestate mapping from the states or prestates of B to the states and prestates of B', respectively. A **top activity** at p_j with $j = 0, \ldots, N$ or at p'_m with $m = 0, \ldots, L$ is an activity of the highest priority among those activites for which at least one directional of the required type is available at p_j or p'_m, in B and B', respectively.

In this section it will be shown that the reduction $p'_0, \ldots, p'_L$ is a realization of the readjustment process in B'. We refer to this as the **realization property** of the EV-reduction. The realization property can be expressed by three conditions (h1) to (h3) listed below:

(h1) The prestate p'_1 results from the prestate p'_0 by the application of a top activity to a directional.

(h2) Let m be one of the numbers $1, \ldots, L-1$. Then p'_m results from p'_{m-1} by the application of an activity h to a directional of the required type for it in B' and p'_{m+1} results from p'_m by the application of an activity k to a directional of the required type for it in B'. Moreover for $h \neq k$ no directional of the required type for activity h is available at p'_m in B' and activity k is the top activity at p'_m in B'.

(h3) The prestate p'_L is saturated in B'

We refer to the three conditions (h1), (h2), and (h3) as the **readjustment rules**. It is clear that these three readjustment rules are equivalent to the statement that $p'_0, \ldots, p'_L$ is a realization of the readjustment process in B'.

We shall sometimes speak of an activity k being applied to a directional in the step from p'_j to p'_{j+1} even if we did not yet show that $p'_0, \ldots, p'_L$ is a realization in B'. We mean by this that p'_{j+1} results from p'_j by the application of the activity to the directional in B'.

6.9.2. Preview. In the following we shall provide an informal account of what will be done in order to prove the realization property of the EV-reduction. The elimination of EV replaces ∂EV_R by the right hand side R of the confluence for ∂EV. Therefore it is of crucial importance that ∂EV_R and R have the same value at every normal prestate in $p_0, \ldots, p_N$. Otherwise a main term could have a different value at a normal prestate p_j in B and at p_j^0 in B'. This would entail a misrepresentation of the substantial relationship modelled by the concerning confluence or restriction equation.

Lemma 31 concerns prestates p_{j+1} following an exceptional state p_j. It is shown that at such a prestate the values of ∂EV_R and R are equal. These prestates are not necessarily normal, but lemma 31 is important for showing by an induction argument that the values of ∂EV_R and R are equal at every normal prestate in $p_0, \ldots, p_N$. This is the content of lemma 32.

Let p_j be a normal prestate and let $m = \tau(j)$ be the renumbered index of p_j^0 in $p'_0, \ldots, p'_L$. It has to be shown that a directional is of the required type for an activity k at p'_m if and only if the same directional is of the required type for the same activity at p'_m. This result, stated by lemma 33, has the consequence that a top activity at p_j is also a top activity at p'_m. Lemma 33 is also important for proving lemma 34 which makes the following statement: If in the step from p_j to p_{j+1}, an activity k is applied to a directional then the application of the same activity to the same directional leads from p'_m to p'_{m+1} in B'.

Theorem 8 states the realization property of the EV-reduction. The proof makes use of lemma 33 and lemma 34.

6.9.3. Derivation of the realization property.

LEMMA 31. *Let p_j be an exceptional prestate in $p_0, \ldots, p_N$. Then at p_{j+1} the value of ∂EV_R is equal to the value of the right hand side R of the confluence for ∂EV in B.*

PROOF. Assume that EV is a source. Then p_0 is the only exceptional prestate and ∂EV is adapted and confirmed in the step from p_0 to p_1. Moreover ∂EV is firm at p_1. Therefore the assertion holds in the source case.

Now assume that EV is a link with the determinator ∂XY. If at p_j one of the activities 1,4, or 5 is applied to ∂EV then in view of (g2) one of these activities has been applied to ∂XY in the step from p_{j-1} to p_j and the assertion holds at p_j.

Suppose that activity 2 is applied to ∂EV in the step from p_j to p_{j+1}. Then (g3) the determinator ∂XY has been dampened in the step from p_{j-1} to p_j. It follows that the assertion holds at p_j.

Suppose that activity 3 is applied to ∂EV in the step from p_j to p_{j+1}. In this case by (g4) we must have $p_j = r(4,1)$ and ∂XY as well as ∂EV must have been dampened between $r(2,1)$ and $r(4,1)$. Therefore at $r(4,1)$ the right tendencies ∂XY_R as well as ∂XY_R are zero at p_j. Therefore the assertion holds in this case, too. This completes the proof of the lemma. □

LEMMA 32. *At every normal prestate p_j in $p_0, \ldots, p_N$ the value of ∂EV_R is equal to the value of the right hand side R of the confluence for ∂EV.*

PROOF. Assume that EV is a source. Then p_0 is exceptional and ∂EV is firm at p_1. Therefore the assertion holds in this case. In the following we shall assume that EV is a link with the determinator ∂XY of ∂EV.

We first show that p_0 is normal. Activities 1, 4, or 5 cannot be applied to ∂EV in the step from p_0 to p_1 since otherwise one of these activites would have to be applied to ∂XY. Similarly activity 2 cannot be applied to ∂EV in this step since

∂XY would have to be dampened before. Activity 3 cannot be applied to ∂EV in the step from p_0 to p_1 since at p_0 the tendency ∂EV is univalued and adjusted in view of (g1). It follows that p_0 is normal.

We are going to use an induction argument. Since p_0 is normal it is sufficient to show that the assertion holds for p_j if it holds for every normal prestate p_k with $k < j$.

Let p_j be a normal prestate with $j > 0$ and assume that the assertion holds for every normal prestate p_k with $k < j$. If p_{j-1} is exceptional then the assertion holds in view of lemma 33. In the following we assume that p_{j-1} is normal.

Consider the case that in the step from p_{j-1} to p_j an activity has been applied to a directional other than ∂XY. The tendency ∂XY_R is not changed in this step and also not ∂EV_R since p_{j-1} is normal. Therefore in this case the assertion holds for p_j since it holds for p_{j-1}. In the following we assume that in the step from p_{j-1} to p_j an activity has been applied to ∂XY.

It is not possible that in the step from p_{j-1} to p_j one of the activities 1, 2, 4, or 5 is applied to ∂XY, since in this case p_j would have to be exceptional by (g2) or (g3). Therefore activity 3 has been applied to ∂XY in the step from p_{j-1} to p_j. The prestate p_{j-1} is normal and the assertion holds for p_{j-1}. The application of activity 3 to ∂XY changes ∂XY_L but neither ∂XY_R nor ∂EV_R. Therefore the assertion holds for p_j. This completes the proof of the lemma. □

REMARK. *The proof of the lemma has shown that p_0 is normal if EV is a link.*

LEMMA 33. *Let p_j with $j = 0, \dots, N-1$ be a normal prestate and let $m = \tau(j)$ be the renumbered index of p_j^0 in $p'_0, \dots, p'_L$. Then a directional other than ∂EV is of the required type for an activity k at p'_m in $B' = M_{EV}(B)$ if and only if it is of the required type for activity k at p_j in B.*

PROOF. Whether a directional is of the required type for an activity k or not depends on whether it is loose or firm, mature or immature, adjusted or maladjusted, and in the case of a tendency, whether it is univalued or split and whether it is a zero tendency or non-zero tendency.

The prestate mapping removes $\partial EV_L, \partial EV_R$ and the confirmation status of ∂EV and leaves everything unchanged. The construction of B' involves a replacement of ∂EV by R in the main terms of confluences and restriction equations and subsequent equivalent transformations of these main terms. At p_j we have $\partial EV_R = R$ in view of lemma 32. Consider the properties, on which it depends whether a directional is of the required type for activity k or not. For each of these properties it follows by what has been said above, that a directional other than ∂EV has this property at p'_m in B' if and only if it has this property at p_j in B. Therefore the assertion of the lemma holds. □

LEMMA 34. *Let p_j with $j = 0, \ldots, N-1$ be a normal prestate and let $m = \tau(j)$ be the renumbered index of p_j^0 in $p'_0, \ldots, p'_L$. If an activity k is applied to a directional in the step from p_j to p_{j+1}, then p'_{m+1} results from p'_m by the application of the same activity to the same directional in B'.*

PROOF. Let p_{j+i} be the next normal prestate after p_j. Since p_N is normal there is such a prestate in $p_0, \ldots, p_N$. We have $m' + 1 = \tau(j+i)$. Let k be the activity applied in the step from p_j to p_{j+1}. If the directional to which activity k is applied is a system specific restriction, then we must have $k = 1$ in view of lemma 1 in 4.5. Since p_j is normal the directional to which activity k is applied in the step from p_j to p_{j+1} cannot be ∂EV.

Suppose that in the step from p_j to p_{j+1} activity k is applied to a tendency ∂WZ. Then the values of ∂WZ_L, ∂WZ_R, or the confirmation status of ∂WZ may be changed in this step but nothing else. By lemma 33 the tendency ∂WZ is of the required type for the application of activity k at p'_m in B'. In view of the definition of the prestate mapping it is clear that p_{j+1}^0 results from $p_j^0 = p'_m$ by the application of activity k to ∂WZ in B'. Essentially the same argument can be used in the case that activity 1 is applied to a system specific restriction $\Box UY$ in the step from p_j to p_{j+1}.

If p_{j+1} is normal then we have $p'_{m+1} = p_{j+1}^0$. The assertion of the lemma holds in this case. Suppose that there are exceptional prestates $p_{j+1}, \ldots, p_{j+i}$ between p_{j+1} and p_{j+i}. Then in the steps from p_{j+1} to p_{j+i} only ∂EV_L, ∂EV_R and the confirmation status of ∂EV can be changed. The prestate mapping π removes these components and changes nothing else. Therefore we have

$$p_{j+1}^0 = p_{j+i}^0 = p'_{m+1}$$

It follows that the assertion of the lemma holds. □

THEOREM 8. *Let EV be an eliminable variable in a base $B = (\Lambda, \Gamma)$ and let $p_0, \ldots, p_N$ be an EV-reducible realization of the readjustment process in B. Moreover let $p'_0, \ldots, p'_L$ be the EV-reduction of $p_0, \ldots, p_N$. Then $p'_0, \ldots, p'_L$ is a realization of the readjustment process in the reduced base B' of B after the elimination of EV.*

COROLLARY 2. *Let s and s' be the states generated by p_N and p'_L in B and B' respectively. Then we have $s' = \lambda(s)$, where λ is the state mapping for the elimination of EV.*

PROOF. We first look at the source case. Assume that EV is a source. Then p_0 is the only exceptional prestate of $p_0, \ldots, p_N$ and EV is adapted and confirmed in the step from p_0 to p_1. If activity k is applied in the step from p_1 to p_2 then we have $k = 1$ or activity k is the top activity at p_1, since $p_0, \ldots, p_N$ is a realization of the readjustment process. Of course, in the case $k = 1$ activity k is also the top activity at p_1. It follows by lemmas 33 and 34 that (h1) is satisfied. Conditions (h2) and (h3) are also consequences of these lemmas together with the fact that $p_0, \dots, p_N$ is a realization in B. Therefore the assertion of the theorem holds in the source case. The corollary will be proven later.

From now on we assume that EV is a link and that ∂XY is the determinator of ∂EV. In view of the remark after the proof of lemma 32 the prestate p_0 is normal. Therefore it follows by lemma 34 that in the steps from p'_0 to p'_1 the same activity is applied to the same directional. Moreover this activity is the top activity at p_0 in B. It follows by lemma 33 that the same activity is the top activity at p'_0 in B'. Therefore (h1) holds.

Since p_N is saturated it follows by the definition of B' together with the definition of the prestate mapping that in view of lemma 32 the prestate p'_L is saturated in B'. Therefore (h3) holds. It remains to show (h2).

As in (h2) let m be one of the integers $1, \dots, L-1$. Let j be the number with $p^0_j = p'_m$. Moreover let i be the number with $p^0_i = p'_{m-1}$. It follows by the definition of $p'_0, \dots, p'_L$ that p_j is normal and that p_i is the last normal prestate before p_j in $p_0, \dots, p_N$. Assume that in the step from p_i to p_{i+1} activity h is applied to a directional $\Box VX$ or ∂VY and that in the step from p_j to p_{j+1} activity k is applied to a directional $\Box UX$ or ∂UY. It follows by lemma 34 that p'_m results from p'_{m-1} by the application of activity h to $\Box VX$ or ∂VY, resp., in B', and that p'_{m+1} results from p'_m by the application of activity k to $\Box UX$ or ∂UY, resp., in B'.

Consider the case $i = j - 1$ in which p_{j-1} is normal. Condition (h2) is satisfied for $h = k$. In the following we assume $h \neq k$. Since $p_0, \dots, p_N$ is a realization of the readjustment process in B it follows that no directional of the required type for activity h is available at p_j and that activity k is the top activity at p_j. It is a consequence of lemma 33, that at p'_m no directional of the required type for activity h is available in B' and that activity k is the top activity at p'_m in B'. Therefore (h2) is satisfied if p_{j-1} is normal. From now on we assume that p_{j-1} is exceptional.

In the following it will be shown that there can be at most two exceptional steps between p_{i+1} and p_j. In the step from p_i to p_{i+1} activity h is applied to a directional other than ∂EV. Let f be the number of the activity applied to ∂EV in the step from p_{i+1} to p_{i+2}. For the case that p_{i+2} is exceptional let g be the number of the activity applied to ∂EV in the step from p_{i+2} to p_{i+3}.

It will also be shown that there are only six possible constellations of the parameters h, f and g. These constellations correspond to the rows of Table 44. If there is only one exceptional step from p_{i+1} to p_j, then a dash is shown in the column for g. Five further columns correspond to the possible values of k.

<table>
<tr><th colspan="3">Steps from p_i to p_j</th><th colspan="5">The step from p_j to p_{j+1}
activity k</th></tr>
<tr><th>h</th><th>f</th><th>g</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr>
<tr><td>1</td><td>1</td><td>–</td><td colspan="5" rowspan="3">1</td></tr>
<tr><td>4</td><td>4</td><td>–</td></tr>
<tr><td>5</td><td>5</td><td>–</td></tr>
<tr><td>2</td><td>3</td><td>–</td><td colspan="2" rowspan="2">2</td><td rowspan="2">3</td><td colspan="2" rowspan="2">4</td></tr>
<tr><td>2</td><td>2</td><td>3</td></tr>
<tr><td>4</td><td>1</td><td>–</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr>
</table>

TABLE 44. Activity number constellations

Table 44 indicates "areas" of activity number constellations. These areas are numbered from 1 to 9. The number of an area is indicated in its upper right corner. Some of these areas are crossed out. As we shall see later the crossed out areas contain impossible activity number constellations. A case distinction based on the nine areas will be used in order to prove that (h2) is satisfied.

We now turn our attention to the seven possibilities for h, f, and g in Table 44. With the help of a case distinction based on the value of f it will be shown that there are no other possibilities and that there cannot be more than two exceptional steps between p_{i+1} and p_j.

Suppose that f has one of the values 1, 4, or 5. After the application of one of these activities to ∂EV this tendency is firm. Moreover, by (g2), immediately before this, one of these activities (not necessarily the same one) is applied to ∂XY. Therefore there is only one exceptional step between p_{i+1} and p_j (?), i.e. we have $j = i + 2$.

We now show that the following statements hold

(i) $h = 1$ implies $f = 1$
(ii) $h = 5$ implies $f = 5$
(iii) $h = 4$ implies $f = 1$ or $f = 4$
(iv) $f = 1$ implies $h = 1$ or $h = 4$
(v) $f = 4$ implies $h = 4$
(vi) $f = 5$ implies $h = 5$.

If ∂XY is adapted and confirmed then ∂EV becomes mature and by (g2) must be adapted and confirmed in the next step. This yields (i).

The flow chart of Figure 8 in 4.5 shows that only activity 5 can be applied to ∂EV after activity 5 has been applied to ∂XY. This yields (ii).

Assume that activity 4 is applied to ∂XY. Then by (g2) the phase of activity 4 either continues with the application of activity 4 to ∂EV or a new phase begins after the confirmation of ∂XY. Since ∂EV has become mature, activity 1 is the top activity at the beginning of the new phase and must be applied to ∂EV. This yields (iii).

Statement (iv) follows by (g2) and (ii). Similarly (v) follows by (g2) together with (i) and (ii). Finally (vi) follows by (g2) together with (i) and (iii).

It can be seen that the four possibilities for $f = 1, 4, 5$ compatible with (iv), (v) or (vi) are covered by Table 44. We now turn our attention to the cases $f = 2$ and $f = 3$.

Assume $f = 2$. In view of (g3) we must have $h = 2$. This means that ∂XY and ∂EV are both dampened. At $r(4, 1)$ both tendencies are split non-zero tendencies. The situation at $r(4, 1)$ is described by case 1 of Table 43 (?). In view of (g1) the tendency ∂EV was adjusted at p_0 but since the value of ∂XY_R changed from $-$ to $+$ to zero, ∂EV is maladjusted at $r(4, 1)$ and therefore is adapted in the step from $r(4, 1)$ to the next prestate. Consequently we have $g = 3$. By lemma 10 a dampened maladjusted tendency cannot become adjusted by later dampenings. Therefore not only ∂EV but also ∂XY is maladjusted at $r(4, 1)$ and must be adapted after ∂EV in the same adaptation phase. It follows that no exceptional step immediately follows the adaptation of ∂EV.

The dampening of ∂EV may be followed by further dampenings or immediately by the adaptation of ∂EV. In the first case we have $f = 3$ and $h = 2$ and there is exactly one exceptional prestate between p_{i+1} and p_j. In the second case we have $h = 2$ as well as $f = 2$ and $g = 3$.

It is now clear that there can be at most two exceptional steps between p_{i+1} and p_j and that Table 44 represents all possibilities with respect to h, f, g, and k. It remains to show that (h2) is satisfied in each of the areas 1 to 9 of Table 44.

In area 1 we have $h = f$. Moreover, in this area there is exactly one exceptional step between p_{i+1} and p_j. Since $p_0, \ldots, p_N$ is a realization of the readjustment process, it follows that for $k \neq f$ no directional of the required type for activity f is available at p_j and that activity k is the top activity at p_j. In view of $h = f$ it follows by lemma 33 and lemma 34 that (h2) is satisfied for the activity number constellations in area 1.

We now examine areas 2, 3, and 4. It has been pointed out above that not only ∂EV but also ∂XY is maladjusted at $r(4, 1)$ if ∂XY and ∂EV have been dampened before. Therefore ∂XY has to be adapted after ∂EV in the same adaptation phase. It follows that only $k = 3$ is possible in the rows with $h = 2$. Accordingly the areas 2 and 4 are crossed out.

Consider a constellation in area 3. Dampening ad adaptation do not change the confirmation status. Therefore no loose directional has become firm in the steps after the end of the initial phase of activity 2 – if there was one – until p_j. The same is true for the steps from p_0 to p_j if there was no such phase. Consequently no directional can have become mature in these steps. It follows that at p_j no directionals of the required type for activity 1 are available. Therefore activity 1 cannot be the top activity at p_j.

Suppose that at p_j a tendency ∂WZ is a maladjusted non-zero tendency. Since the adaptation of ∂EV at $r(4, 1)$ does not change anything else than the value of ∂EV_L the tendency ∂WZ must have been a maladjusted non-zero tendency at $r(4, 1)$. In this case ∂WZ would have to be dampened in the step from $r(4, 1)$ to the next prestate contrary to the definition of $r(4, 1)$. Therefore activity 2 cannot be the top activity at p_j. It follows that activity 3 is the top activity at p_j. In view of lemma 33 and lemma 34 this yields the conclusion that (h2) is satisfied in area 3.

Now assume $h = 4$ and $f = 1$. In this case activity 4 has been applied to ∂XY at the end of a phase of activity 4 between $r(6, m)$ and $r(8, m)$. In the step from $r(8, m)$ to the next prestate ∂EV is adapted and confirmed. This next prestate is the prestate p_j. The flow chart of Figure 8 shows that we have $k = 1$ if at p_j the answer to the question of switch 10 is YES. If this answer is NO and the answer to the question of switch 12 is YES, then at least one maladjusted tendency must be adapted at rectangle 7. In this case we have $k = 3$. It is also possible that at p_j the answer to the question of switch 10 as well as to the question of switch 12 is NO. In this case we have $k = 5$. It is clear that we cannot have $k = 2$ or $k = 4$. Therefore areas 6 and 8 are crossed out.

At $r(6, m)$ all tendencies are adjusted. It is a consequence of lemma 4 that an adjusted non-zero tendency remains an adjusted non-zero tendency if activity 4 is applied to another tendency. At $r(8, m)$ all non-zero tendencies are adjusted and firm and all maladjusted tendencies including ∂EV are zero tendencies. This is not changed by adaptation and confirmation of ∂EV in the step from $r(8, m)$ to p_j. Therefore at p_j no directionals of the required type for activity 4 are available. It follows by lemma 33 that at p'_m no directionals of the required type for activity 4 in B' are available. Consequently (h2) is satisfied in the areas 5, 7, and 9 if k is the top activity at p'_m in B'. In view of lemma 33 this is true if activity k is the top activity at p_j. It remains to show that this is the case.

Obviously activity 1 is the top activity at p_j for $k = 1$. In the following assume $k \neq 1$. In this case the phase of activity 1 beginning at $r(8, m)$ ends at p_j and there a phase of activity k begins. Since $p_0, \dots, p_N$ is a realization of the readjustment process, activity k must be the top activity at p_j. Therefore (h2) holds.

It remains to prove the corollary. We have $p_N = p_0(s_1)$ and $p'_L = p'_0(s'_1)$. In view of statement (2) of lemma 29 in 6.6 it follows by $p'_L = \pi(p_N)$ that we have $s'_1 = \lambda(s_1)$. This completes the proof of the theorem including its corollary. □

6.10. Invariance of the transition diagram

6.10.1. Definition of invariance of the transition diagram. We continue to use the notational conventions introduced in 6.7.1 and 6.9.1. In 6.7.1 it has been explained what it means that the result of a transition cause ω at s is invariant under the state mapping. Lemma 37 will show that the result of a main transition cause ω at s is invariant under the state mapping.

In this section and the following one we shall look at transition diagrams and extended transition diagrams. The definition of these diagrams involves the priority order ρ and the perturbance assignment α. Therefore it is not sufficient to talk about a base $B = (\Lambda, \Gamma)$ and its modifications. It is necessary to deal with full qualitative dynamic systems. In the remainder of this chapter $\Phi = (\Lambda, \Gamma, \rho, \alpha)$ will always be an arbitrary but fixed qualitative system with the base $B = (\Lambda, \Gamma)$ with at least one removable variable. Moreover RV will be a removable variable of Φ.

The definition of the reduced system $\Phi' = (\Lambda', \Gamma', \rho', \alpha')$ of Φ after the elimination of RV has been introduced in 6.5. In the remainder of this chapter $B' = (\Lambda', \Gamma')$ is the base of Φ'. Lemma 26 in 6.4 has shown that the state mapping λ is a one-to-one mapping of the set of all states for B onto the set of all states for B'. It follows by lemma 27 that a main transition cause ω is invariant under the state mapping, i.e. it is pending at $\lambda(s)$ if and only if it is pending at s. In view of the definition of the reduced priority ranking ρ' of Φ' we have

$$\rho'(\omega, s') = \rho(\omega, s) \quad \text{for } s' = \lambda(s)$$

if ω is a main transition cause pending at s.

We say that the tentative transition diagram of Φ is **invariant under the elimination of** RV, if for every main transition cause ω of positive rank $\rho(\omega, s)$ pending at a state s for B the result of ω at s is invariant under the state mapping in the sense of 6.7.1.

Let k^* be the rank of the transition diagram of Φ. The transition diagram of Φ' is **invariant under the elimination of** RV if the transition diagram of Φ' has the same rank k^* and in addition to that every main transition cause ω with

$$0 < \rho(\omega, s) \leq k^*$$

pending at a state s for B the result of ω at s is invariant under the state mapping λ.

Lemma 35 will show that for a shift or a lag extinction ω pending at a state s the result of ω at s is invariant under the state mapping. This result will be extended to all main transitions. With the help of these lemmas it can then be proven that the tentative transition diagram and the transition diagram are invariant under the elimination of RV. This will be the content of theorem 9.

Comment. *The invariance of the tentative transition diagram means that in the transition from Φ to Φ' the graph structure is not changed. A node which represents a state s in the diagram for Φ, represents the state $\lambda(s)$ in the diagram for Φ'. Nothing else is different. An edge represents the same transition cause in the two diagrams, and by the definition of the reduced priority ranking ρ', the rank of a transition cause also remains the same one.*

What has been said about the tentative transition diagrams, also holds for the transition diagrams of Φ and Φ'. The rank k^* is the same one for the two transition diagrams.

6.10.2. Derivation of invariance results.

Lemma 35. *Let ω be a shift or a lag extinction pending at a state s of B. Then the result of ω at s is invariant under the state mapping λ.*

Proof. In view of the removability conditions (e2) and (e4) the variable RV is unscaled and lag free. Therefore a shift must be the shift of another variable and a lag extinction must concern a lagged tendency of another variable.

Let $p_0 = p_0(\omega, s)$ be the transition start for ω at s. In view of statement (1) in lemma 30 an RV-reducible realization beginning with p_0 exists. Let $p_0, \ldots, p_N$ be such a realization and let $p'_0, \ldots, p'_L$ be the RV-reduction of $p_0, \ldots, p_N$. Moreover let s_1 and s'_1 be the states for B and B' generated by p_N and p'_L, respectively. In view of the corollary of theorem 8 we have $s'_1 = \lambda(s_1)$. Therefore the result $s_1 = z(\omega, s)$ of ω at s is invariant under the state mapping. This completes the proof of the lemma. □

LEMMA 36. *Let B be a base with a removable variable RV and let B' be the RV-reduction $M_{RV}(B)$ of B. Let ω be a modifier, B_ω be the modified base $M_\omega(B)$ and B'_ω the RV-reduction $M_{RV}(B_\omega)$ of B_ω. Moreover let s be a state for B, let $p_0, \ldots, p_N$ be an RV-reducible realization of the readjustment process in B_ω beginning with $p_0 = p_0(s)$, and let $p'_0, \ldots, p'_L$ be the RV-reduction of $p_0, \ldots, p_N$. Then the following statements (1) and (2) are true:*

(1) *Let $s' = \lambda(s)$ be the image of s under the state mapping λ from the states of B to the states of B'. Then p'_0 is the prestate $p'_0(s')$ of s' in B'.*
(2) *If and only if the state s_1 generated by p_N in B_ω is also a state for B, the state s'_1 generated by p'_L in B_ω is a state for B'.*

PROOF. For the purpose of proving (1) we distinguish between the source case and the link case. Let RV be a link. Then it follows by the remark after the proof of lemma 32 that p_0 is normal. Therefore we have $p'_0 = \pi(p_0(s))$ in the link case. In view of statement (2) of lemma 9 this yields $p'_0 = p'_0(s')$.

Now assume that RV is a source. In this case p_0 is exceptional. ∂EV is adapted and confirmed in the step from p_0 to p_1. The prestate mapping from the prestates of B_ω to the prestates of B'_ω is not different from the prestate mapping π from the prestates of B to the prestates of B'. In both cases the specifications of $\partial RV_L, \partial RV_R$, and the confirmation status of ∂RV are deleted and nothing else is changed. Therefore the definition of the RV-reduction for the source case yields $p'_0 = \pi(p_1)$. However, adaptation and confirmation does not change anything else than the specifications which are deleted by the prestate mapping. Therefore we also have $p'_0(\pi(p_0(s))$. As in the link case this yields $p'_0 = p'_0(s')$ in view of statement (2) of lemma 29. Consequently (1) holds.

Let λ_ω be the state mapping from the states for B_ω to the states for B'_ω. The sets of states are different in B and B_ω. Therefore the mappings λ_ω and λ are different. However, if s_1 is not only a state of B_ω but also of B then we have $\lambda(s_1) = \lambda_\omega(s_1)$, since λ and λ_ω delete the same specifications of a state. If s'_1 is not only a state for B'_ω but also for B', then $\lambda^{-1}(s'_1) = \lambda_\omega^{-1}(s'_1)$ must hold since the confluence for ∂RV is the same one in B and B_ω. Therefore (2) holds. This completes the proof of the lemma. □

REMARK. *The proof of lemma 36 has shown that we have $\lambda(s_1) = \lambda_\omega(s_1)$ if s_1 is a state of B_ω and a state of B.*

LEMMA 37. *let ω be a main transition cause pending at a state s for B. Then the result of ω at s is invariant under the state mapping.*

PROOF. By lemma 35 the assertion is true, if ω is a shift or a lag extinction. It remains to show that the assertion holds if ω is a tendency switch.

Let $\omega = [\partial XY \to d]$ be a tendency switch pending at a state s for B. In view of the removability condition (e5) the confluence for ∂RV is monocausal. Therefore a tendency switch of ∂RV is impossible. Consequently ∂RV is not the tendency ∂XY. (However, ∂XY may be the determinator of ∂RV, if RV is a link)

The transition start $p(\omega, s)$ for ω at s is the prestate $p_0(s)$ of s. The prestate $p_0(s)$ is the beginning of a readjustment process in the hypothetical base B_ω for ω. According to statement (2) of lemma 30 an RV-reducible realization in B_ω beginning with $p_0 = p_0(s)$ exists. Let $p_0, \ldots, p_N$ be such an RV-reducible realization in B_ω and let $p'_0, \ldots, p'_L$ be the RV-reduction of $p_0, \ldots, p_N$. In view of statement (1) of lemma 36 we have

$$p'_0 = p_0(s') \text{ with } s' = \lambda(s)$$

According to lemma 27 transition causes are invariant with respect to the state mapping. Therefore ω is pending at $s' = \lambda(s)$ in B'. The transition start $p_0(\omega, s')$ for ω at s' is the prestate $p'_0(s')$ of s' in B'.

Lemma 28 shows that M_{RV} and M_ω commute. Consequently we have

$$B'_\omega = M_{RV}(B_\omega) = M_\omega(B')$$

The base B'_ω is not only the RV-reduction of B_ω but also the hypothetical base for the tendency switch ω at s' in B'. In order to find out whether ω is feasible at s' in B', one has to run the readjustment process in B'_ω beginning with $p'_0(s')$. In view of statement (1) of lemma 36 the RV-reduction $p'_0, \ldots, p'_L$ is such a realization. The tendency switch ω is feasible, if a state of B is generated by p'_L in B'_ω.

Let s_1 be the state for B'_ω generated by p_N and let s'_1 be the state generated by p'_L in B'_ω. The tendency switch ω is feasible at s in B, if s_1 is not only a state of B_ω but also a state of B. Similarly ω is feasible at s' in B' if s'_1 is not only a state of B'_ω but also of B'. It follows by statement (2) of lemma 36 that ω is feasible at s in B if and only if ω is feasible at s' in B'.

As in the proof of lemma 36 let λ_ω be the state mapping from the states of B_ω to the states of B'_ω. The corollary of Theorem 8 applied to B_ω and B'_ω instead of B and B' yields the conclusion that $s'_1 = \lambda_\omega(s_1)$ holds. In view of the remark after the proof of lemma 36 we have $s'_1 = \lambda(s_1)$ if ω is feasible at s. Since in this case s_1 is the result $z(\omega, s)$ of ω at s and s'_1 is the result $z'(\omega, s')$ of ω at s' we can conclude that a feasible tendency switch is invariant under the state mapping.

It remains to show that the assertion holds if ω is semifeasible or infeasible. Assume that $\omega = [\partial XY \to d]$ is not feasible. Then ω must be a tardy tendency switch, since by theorem 4 immediate tendency switches are always feasible. We have $d \neq 0$ and at the state s the value of ∂XY is $-d$.

In order to find out whether ω is semifeasible or infeasible one has to examine the halfway switch $\mu = [\partial XY \to 0]$. The conclusion that a semifeasible tendency switch is invariant under the state mapping can be reached in the same way as the analogous conclusion about feasible tendency switches derived above. The hypothetical base B_μ and its RV-reduction B'_μ take the place of B_ω and B'_ω but otherwise almost literally the same arguments apply.

As we have seen above ω is feasible at s if and only if ω is feasible at s' in B'. In the same way we can derive the following statement: A tendency switch ω which is not feasible at s is semifeasible at s if and only if ω is semifeasible at $s' = \lambda(s)$ in B. It follows that ω is infeasible at s if and only if ω is infeasible at s' in B'. Moreover we have

$$z'(\omega, \lambda(s)) = \lambda(z(\omega, s))$$

if ω is semifeasible at s. We can conclude that the result of ω is invariant under the state mapping, if ω is semifeasible or infeasible at s. This completes the proof of the lemma. □

THEOREM 9. *Let $\Phi = (\Lambda, \Gamma, \rho, \alpha)$ be a qualitative dynamic system and let RV be a removable variable for Φ. Then the tentative transition diagram of Φ as well as the transition diagram of Φ is invariant under the elimination of RV.*

PROOF. The invariance of the tentative transition diagram is an immediate consequence of lemma 37. Let ω be a main transition cause at a state s and let $s' = \lambda(s)$ be the image of s under the state mapping. In view of the definition of the reduced priority ranking ρ' after the elimination of RV we have

$$\rho'(\omega, \lambda(s)) = \rho(\omega, s)$$

Consider a sequence $s_1, s_2, \ldots$ of states for Φ and let $s'_1, s'_2, \ldots$ with $s'_i = \lambda(s_i)$ be the sequence of the images of $s_1, s_2, \ldots$ under the state mapping. It is clear that $s'_1, s'_2, \ldots$ is a tentative path for Φ' if and only if $s_1, s_2, \ldots$ is a tentative path for Φ. It can also be seen that $s'_1, s'_2, \ldots$ has an unresolved shift or an unresolved lag extinction, if and only if $s_1, s_2, \ldots$ has an unresolved shift or an unresolved lag extinction (se 3.10). In this respect it is important that in view of (e2) and (e4) there cannot be any shifts of RV or lag extinctions of ∂RV^-. It follows that $s'_1, s'_2, \ldots$ is a permissible path for Φ' if and only if $s_1, s_2, \ldots$ is a permissible path for Φ. Moreover, the rank of $s'_1, s'_2, \ldots$ is equal to the rank of$s'_1, s'_2, \ldots$. It follows that the tentative transition diagram of Φ is well structured, if and only if the tentative transition diagram of Φ' is well structured. By theorem 6 in 5.7 the tentative transition diagram of Φ is well structured and therefore the tentative transition diagram of Φ' is well structured, too. It can also be seen that the rank k^* of the tentative transition diagram of Φ is also the rank of the tentative transition diagram of Φ'. Consequently the transition diagram of Φ is invariant under the elimination of RV. This completes the proof of the theorem. □

6.11. Stability invariance

6.11.1. Invariance of notions connected to stability. The use of notational conventions introduced in 6.7.1, 6.9.1, and 6.10.1 is continued. In 6.7.1 it has been explained what it means that the result of a perturbance ω at a potentially stationary state s for Φ is invariant under the state mapping.

We say that stationarity in Φ is **invariant under the elimination** of RV if the following is true: $s' = \lambda(s)$ is stationary in Φ' if and only if s is stationary in Φ. The extended transition diagram is called **invariant under the elimination** of RV if the following conditions (i1), (i2), and (i3) are satisfied:

(i1) The transition diagram of Φ is invariant under the elimination of RV.
(i2) Stationarity in Φ is invariant under the elimination of RV.
(i3) At every stationary state s for Φ and for every expected perturbance $\omega \in \alpha(s)$, the result of ω at s is invariant under the state mapping.

In 5.9 several concepts related to stability and instability have been introduced. Definitions have been given for the seven terms shown in the fields of Table 45. These notions have been defined by conditions on permissible paths in the transition diagram starting from reentry states.

	Properties of	
	ω and s *)	s alone *)
Instability properties	destabilizable escapable unreachable	unstable repulsor
Stability properties		stable recaptor

TABLE 45. Seven stability and instability properties

*) Here s is a stationary state and $\omega \in \alpha(s)$ is an expected perturbance at s, where α is the perturbance assignment of Φ.

Some of the terms in Table 45 express properties of a perturbance ω and a stationary state s and others stand for properties of a stationary state s alone.

Some of the notions expressed by these terms are **stability properties** and others are **instability properties**. Table 45 exhibits these distinctions. All properties of ω at s are instability properties. Therefore the lower left field of Table 45 is empty.

Each of the properties of ω and s in Table 45 is **invariant under the elimination of** RV if the following is true: The perturbance ω has this property at $s' = \lambda(s)$ in Φ' if and only if it has it at s in Φ. This definition presupposes invariance of stationarity in Φ. Similarly a property of s in Table 45 is **invariant under the elimination** of RV if the following is true: The stationary state $s' = \lambda(s)$ has this property in Φ' if and only if s has it in Φ.

It is the goal of this section to derive the invariance of the extended transition diagram and of the seven stability and instability properties under the elimination of RV. This will be the content of Theorem 10. As a first step towards this goal lemm 38 will establish the invariance of stationarity.

For the purpose of proving theorem 10 we need the concept of the **immediate transition diagram** of an auxiliary base B_ω of B. This diagram is a directed graph with an immediate transition cause attached to each edge. The nodes represent the states of B_ω. The edges represent immediate transitions. The direction of an edge goes from a state u to the transition result v of the immediate transition cause μ attached to the edge. The diagram represents all immediate transitions pending at a state.

Consider an auxiliary base B_ω of B for a perturbance ω of a tendency other than ∂RV. It is clear that RV is eliminable in B_ω. Let B'_ω be the reduction of B_ω after the elimination of RV. We say that the immediate transition diagram of B_ω is **invariant under the elimination** of RV, if the immediate transition diagram for B'_ω is the same one as that for B_ω with the only difference that a node, which represents u in the diagram of B_ω, represents $\lambda_\omega(u)$ in the diagram for B'_ω. Here λ_ω is the state mapping of B_ω under the elimination of RV. The invariance of B_ω under the elimination of B_ω will be established by lemma 39.

Another concept which will be needed is that of "parallel" reentry histories for Φ and Φ'. The notion of a reentry history has been described by Table 26 in 5.8. A **reentry history in** Φ is a sequence

$$s, \omega, p_0, q_0, a_0, \dots, a_M, q, p, e$$

in which s is a stationary state of Φ and ω is an expected perturbance $\omega \in \alpha(s)$ at s. In addition to this we have $p_0 = p_0(s)$ and $q_0 = h_\omega(p_0)$ as well as $a_0 = g(q_0)$. Moreover $a_0, \dots, a_M$ is an immediate transition chain for B_ω, we have $q = p_0(a_M)$ and $p = h(q)$ as well as $e = g(p)$. A reentry history

$$s', \omega', p'_0, q'_0, a'_0, \dots, a'_M, q', p', e'$$

in Φ' is defined analogously.

We say that two reentry histories

$$s', \omega', p'_0, q'_0, a'_0, \ldots, a'_M, q', p', e'$$

in Φ' and

$$s, \omega, p_0, q_0, a_0, \ldots, a_M, q, p, e$$

in Φ are **parallel** to each other if the following four **parallelity conditions** are satisfied:

(j1) $s' = \lambda(s)$
(j2) $\omega' = \omega$
(j3) $a'_m = \lambda_\omega(a_m)$ for $m = 1, \ldots, M$
(j4) $e' = \lambda(e)$

Here λ_ω is the state mapping for B_ω under the elimination of RV. It will be shown that for every reentry history in Φ there is exactly one parallel reentry history in Φ' and vice versa. This will be the content of lemma 40.

The term "stability invariance" is meant to include all the invariance notions connected to stability introduced above, the invariance of the extended transition diagram and the invariance of the seven stability and instability properties in Table 45. However, this term will only be used informally.

6.11.2. Derivation of stability invariance.

LEMMA 38. *Let Φ be a qualitative dynamic system and let RV be a removable variable of Φ. Then stationarity in Φ is invariant under the elimination of RV.*

PROOF. As before let B be the base of Φ and let B' be the reduction of B after the elimination of RV. Moreover let λ be the state mapping for B under the elimination of RV. In view of the remark after lemma 27 a state $s' = \lambda(s)$ for B' is potentially stationary in Φ' if and only if s is potentially stationary in Φ.

In 3.6 a stationary state s has been defined as a potentially stationary state with the additional property that $\phi_1(s)$ is empty or contains no other main transition causes other than infeasible tardy tendency switches. Main transition causes are invariant under the state mapping according to lemma 27 and their results are invariant under the state mapping by lemma 37. In view of the definition of the reduced priority ranking ρ' for B' in 6.5 it follows that the assertion holds. This completes the proof of the lemma. □

LEMMA 39. *Let B_ω be an auxiliary base of the base B of Φ for a perturbance ω other than ∂RV. Then the immediate transition diagram of B_ω is invariant under the elimination of RV.*

PROOF. Let B'_ω be the reduction of B_ω after the elimination of RV and let λ_ω be the state mapping of B_ω under the elimination of RV. The state mapping λ_ω is a one-to-one mapping from the set of all states for B_ω onto the set of all states of B'_ω. The immediate transition causes represented by the edges of the immediate transition diagram are main transitions. Therefore the invariance of the immediate transition diagram of B_ω is an immediate consequence of lemma 37, applied to B_ω instead of B. This completes the proof of the lemma. □

LEMMA 40. *let Φ be a qualitative dynamic system, let RV be a removable variable in Φ and let Φ' be the reduction of Φ after the elimination of RV. Then there is exactly one parallel reentry history in Φ' for every reentry history in Φ. Similarly there is exactly one parallel reentry history in Φ for every reentry history in Φ'.*

PROOF. Consider a reentry history

$$s, \omega, p_0, q_0, a_0, \ldots, a_M, q, p, e$$

It will now be shown that there is exactly one reentry history

$$s', \omega, p'_0, q'_0, a'_0, \ldots, a'_M, q', p', e'$$

such that the four parallelity conditions are satisfied. It follows by statement (2) of lemma 29 that we have

$$p'_0(s') = \pi(p_0(s))$$

It is a consequence of lemma 30 that an RV-reducible realization of the readjustment process in the auxiliary base beginning with $p_0 = p_0(s)$ exists. It follows by the corollary of Theorem 8 together with Theorem 3 that a readjustment process in the RV-reduction B'_ω of B_ω beginning with $p'_0(s')$ leads to $a'_0 = \lambda_\omega(a_0)$. For $m = 1, \ldots, M$ define $a'_m = \lambda(a_m)$. Since $a_0, \ldots, a_m$ is an immediate transition chain in B_ω it is a consequence of lemma 39 that $a'_0, \ldots, a'_M$ is an immediate transition chain for B'_ω.

The set of prestates is the same one for B and B_ω. The same is true for B' and B'_ω. The prestate mapping π maps prestates of B to prestates of B' and it also maps prestates of B_ω to prestates of B'_ω. In both cases the specifications of ∂RV_L and ∂RV_R and the confirmation status of ∂RV are taken out and nothing else is changed. We have $q = p_0(a_M)$. This means that right and left tendencies in q have the same value as in a_M and the confirmation status of every directional in q is L. There are no other differences between q and $p_0(a_M)$. This means that right and left tendencies in q have the same value as in a_M and the confirmation status of every directional in q is L. There are no other differences between q and $p_0(a_M)$. The relationship between $q' = p'_0(a'_M)$ and a'_M is analogous. In view of $a'_M = \lambda_\omega(a_M)$ the state a'_M results from a_M by taking out the specification of ∂RV

without changing anything else. It can be seen easily that the same end result is obtained if on the one hand first $a'_M = \lambda_\omega(a_M)$ is formed and then $p'_0(a'_M)$ or on the other hand first $q = p_0(a_M)$ and then $\pi(q)$. Therefore we have

$$q' = \pi(q)$$

In view of statement (3) of lemma 30 an RV-reducible realization of the readjustment process running in B and beginning with q exists. It follows by the corollary of Theorem 8 together with Theorem 3 that a readjustment process in the RV-reduction B' of B beginning with q' leads to $p' = \pi(p)$ and from there to $e' = \lambda(e)$.

We have constructed a reentry history

$$s', \omega, p'_0, q'_0, a'_0, \ldots, a'_M, q', p', e'$$

for B' which satisfies the four parallelity conditions. The construction also shows that there is no other reentry history for B' which is parallel to the reentry history $s, \omega, p_0, q_0, a_0, \ldots, a_M, q, p, e$. As far as s and ω is concerned this follows by (j1) and (j2). The definition of a reentry history then determines p'_0 and a'_0. Condition (j3) requires $a'_m = \lambda(a_m)$ for $m = 1, \ldots, M$. Finally q', p' and e' are determined by the definition of a reentry history.

It remains to show that there is exactly one parallel reentry history in Φ for every reentry history. Consider an arbitrary reentry history

$$s', \omega, p'_0, q'_0, a'_0, \ldots, a'_M, q', p', e'$$

for B'. Since λ and λ_ω are one-to-one mappings onto the set of states for B' and B'_ω, respectively, there is exactly one state s such that (j1) is satisfied and for each $m = 0, \ldots, M$ there is exactly one state a_m for B_ω such that (j3) is satisfied. The other elements of the reentry history

$$s', \omega, p'_0, q'_0, a'_0, \ldots, a'_M, q', p', e'$$

for B are determined by the definition of a reentry history. It is also clear that (j4) holds and that there cannot be any other reentry history for B which is parallel to the reentry history for B'. This completes the proof of the lemma. □

THEOREM 10. *Let $\Phi = (\Lambda, \Gamma, \rho, \alpha)$ be a qualitative dynamic system and let RV be a removable variable for Φ. Then the extended transition diagram of Φ is invariant under the elimination of RV. Moreover, the seven stability and instability properties in Table 45 are invariant under the elimination of RV in Φ.*

PROOF. In order to prove the invariance of the extended transition diagram of Φ under the elimination of RV, we have to show that the conditions (i1), (i2), and (i3) in the definition of this invariance are satisfied. Condition (i1) is a consequence of Theorem 9. In view of lemma 38 condition (i2) is satisfied.

Condition (i3) requires that at every stationary state s the result of a perturbance $\omega \in \alpha(s)$ at s is invariant under the state mapping. By definition this is the case if $E'(\omega, s')$ with $s' = \lambda(s)$ is the set of all $e' = \lambda(e)$ with $e \in E(\omega, s)$.

In view of lemma 40, for every reentry history in B, we can find a parallel reentry history in B'. Therefore $E'(\omega, s')$ contains all $e' = \lambda(e)$ with $e \in E(\omega, s)$. Since for every reentry history in B' we can find a parallel reentry history in B, it also holds that $E'(\omega, s')$ does not contain any e' which is not an image of an $e \in E(\omega, s)$ under the state mapping. It follows that $E'(\omega, s')$ is the set of all $e' = \lambda(e)$ with $e \in E(\omega, s)$. Therefore (i3) is satisfied. Consequently, the extended transition diagram is invariant under the elimination of RV.

It remains to show that the seven properties of stability and instability in Table 45 are invariant under the elimination of RV. Consider first the three properties of ω and s. These properties are defined in terms of the availability or unavailability of certain kinds of permissible paths in the transition diagram beginning with a reentry state $e \in E(\omega, s)$. Destabilizability means that there is at least one such path with at most one tardy transition which does not lead back to s. Escapability means that there is at least one such path which never comes back to s. Unreachability after ω means that every path of this kind never comes back to s. It can be seen immediately that the invariance of these three properties under the elimination of RV is a consequence of the invariance of the extended transition diagram under the elimination of RV.

The other four properties of stability and instability are defined in terms of the first three properties. Being stable means not being destabilizable by any $\omega \in \alpha(s)$ and being instable means not being stable. A repulsor is defined as being unreachable after every $\omega \in \alpha(s)$ and a recaptor is defined as not being escapable by any $\omega \in \alpha(s)$. Here, too, it can be seen immeditely that the invariance of each of these four properties under the elimination of RV is a consequence of the invariance of the extended transition diagram under the elimination of RV. This completes the proof of the theorem. □

6.11.3. Successive elimination. The invariance of the transition diagram and the extended transition diagram under the elimination of a removable variable RV facilitates the analysis of qualitative dynamic systems. As far as the investigation of cycles and properties of stability and instability is concerned a qualitative dynamic system can be replaced by its reduction after the elimination of a removable variable. In some cases the number of variables can be reduced considerably by the successive elimination of removable variables. This simplifies the determination of a list of all possible states and diminuishes the size of prestates and the length of readjustment processes.

Just as well as the elimination of one removable variable, the successive elimination of removable variables leaves the transition diagram and the extended transition essentially unchanged. One can look at a state mapping as the replacement of a longer description of a state by a shorter one. In this sense only the names of the vertices are changed by the elimination of a removable variable. Otherwise the structure of the transition diagram or an extended transition diagram remains the same one. This is also true for the end result after the successive elimination of removable variables.

It has already been pointed out in 6.1 that in the model of Hume's specie flow mechanism (see 2.1) the variables DE, PR, IM, and EX can be successively removed. At the end the base of the system is reduced to the following two confluences:

$$\begin{aligned} \partial GO &= \begin{cases} - & \text{for } TR = D \\ 0 & \text{for } TR = b \\ + & \text{for } TR = S \end{cases} \\ \partial TR &= -\partial GO \end{aligned}$$

Removable variables may have an important role in the interpretation of a system, but they are not needed for the analysis.

Bibliography

[1] Bobrow, D. G. (Ed.), *Qualitative Reasoning about Physical Systems* New York: North-Holland, 1984.

[2] de Kleer, J. and Brown, J.S. (1984), A qualitative physics based on confluences, *Artificial Intelligence* 24 (1-3) pp. 7-83, Reprinted in D. G. Bobrow (Ed.), *Qualitative Reasoning about Physical Systems* New York: North-Holland, 1984.

[3] Faltings, B. and Struss P. (Eds) (1992), *Recent Advances in Qualitative Physics* MIT Press, Cambridge, MA

[4] Fishwick, P. A. and P. A. Luker, P. A., eds. (1991), *Qualitative Simulation Modeling and Analysis*, Springer Verlag

[5] Haberler, G. (1956) [1937], *Prosperity and Depression: A Theoretical Analysis of Cyclical Movements*, London: George Allen and Unwin

[6] Hume, D. (1752), Of the Balance of Payments, in R. N. Cooper (ed), *International Finance, Selected Readings*, Harmondsworth, Baltimore: Penguin Books (1969)

[7] Kuipers, B.J. (1984), Commonsense reasoning about causality: deriving behavior from structure, *Artificial Intelligence* 24: 169-203. Reprinted in D. G. Bobrow (Ed.), *Qualitative Reasoning about Physical Systems* New York: North-Holland, 1984.